

Structure of root graded Lie algebras

Malihe Yousofzadeh¹

Abstract. We give a complete description of Lie algebras graded by an infinite irreducible locally finite root system.

1. INTRODUCTION

In 1992, S. Berman and R. Moody [6] introduced the notion of a Lie algebra graded by an irreducible reduced finite root system. Their definition was motivated by a construction appearing in the classification of finite dimensional simple Lie algebras containing nonzero toral subalgebras [12]. The classification of root graded Lie algebras in the sense of S. Berman and R. Moody was given, in part, by S. Berman and R. Moody themselves and was completed by G. Benkart and E. Zelmanov [5] in 1996. This classification has been based on a type-by-type approach; for each type X , the authors give a recognition theorem for centerless Lie algebras graded by a root system of type X . In 1996, E. Neher [11] generalized the notion of root graded Lie algebras by switching from fields of characteristic zero to rings containing $1/6$ and working with locally finite root systems instead of finite root systems. Roughly speaking, according to him, a Lie algebra \mathcal{L} over a ring containing $1/6$ is graded by a reduced locally finite root system R if \mathcal{L} is a $Q(R)$ -graded Lie algebra generated by homogenous submodules of nonzero degrees and that for any nonzero root $\alpha \in R$, there are homogenous elements e and f of degrees α and $-\alpha$ respectively such that $[e, f]$ acts diagonally on \mathcal{L} . He realized root graded Lie algebras for reduced types other than F_4 , G_2 and E_8 as central extensions of Tits-Kantor-Koecher algebras of certain Jordan pairs. Finally in 2002, B. Allison, G. Benkart and Y. Gao [3] defined a Lie algebra graded by an irreducible finite root system of type BC and studied root graded Lie algebras of type BC_n for $n \geq 2$. In 2003, G. Benkart and O. Smirnov [7] studied Lie algebras graded by a finite root system of type BC_1 and finalized the classification of Lie algebras graded by an irreducible finite root system.

A Lie algebra \mathcal{L} graded by an irreducible finite root system R has a weight space decomposition with respect to a splitting Cartan subalgebra of a finite dimensional split simple Lie subalgebra \mathfrak{g} of \mathcal{L} , whose set of weights is contained in R . This feature allows us to decompose \mathcal{L} as $\mathcal{L} = \mathcal{M}_1 \oplus \mathcal{M}_2$ in which \mathcal{M}_1 is a direct sum of finite dimensional irreducible nontrivial \mathfrak{g} -submodules and \mathcal{M}_2 is a trivial \mathfrak{g} -submodule of \mathcal{L} . One can derive a specific vector space \mathfrak{b} from the \mathfrak{g} -module structure of \mathcal{M}_1 . This vector space is equipped with an algebraic structure which is induced by the Lie

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algebraic structure of \mathcal{L} . Moreover the Lie algebra \mathcal{L} can be reconstructed from the algebra \mathfrak{b} in a prescribed way, see [2] and [3]. This construction led to finding a finite presentation for the universal central extension of a Lie torus of a finite type other than A and C , see [14], [4]. This motivates us to generalize this construction for Lie algebras graded by infinite root systems.

We give a complete description of the structure of root graded Lie algebras. We fix an infinite irreducible locally finite root system R and show that a Lie algebra \mathcal{L} graded by R can be described in terms of a locally finite split simple Lie subalgebra \mathcal{G} , some natural representations of \mathcal{G} and a certain algebra called the *coordinate algebra*. We also give the Lie bracket on \mathcal{L} in terms of the Lie bracket on \mathcal{G} , the action of the representations and the product on \mathfrak{b} . More precisely, depending on type of R , we consider a quadruple \mathfrak{c} so called *coordinate quadruple*. We next correspond to \mathfrak{c} , a specific algebra $\mathfrak{b}_{\mathfrak{c}}$ and a specific Lie algebra $\{\mathfrak{b}_{\mathfrak{c}}, \mathfrak{b}_{\mathfrak{c}}\}$. Then for each subspace \mathcal{K} of the center of $\{\mathfrak{b}_{\mathfrak{c}}, \mathfrak{b}_{\mathfrak{c}}\}$ satisfying a certain property called *the uniform property*, we define a Lie algebra $\mathcal{L}(\mathfrak{b}_{\mathfrak{c}}, \mathcal{K})$ and show that it is a Lie algebra graded by R . Conversely, given a Lie algebra \mathcal{L} graded by R , we prove that \mathcal{L} can be decomposed as $\mathcal{M}_1 \oplus \mathcal{M}_2$ where \mathcal{M}_1 is a direct sum of certain irreducible nontrivial \mathfrak{g} -submodules for a locally finite split simple Lie subalgebra \mathfrak{g} of \mathcal{L} and \mathcal{M}_2 is a specific subalgebra of \mathcal{L} . We derive a quadruple \mathfrak{c} from the \mathfrak{g} -module structure of \mathcal{M}_1 and show that it is a coordinate quadruple. We also prove that there is a subspace \mathcal{K} of $\{\mathfrak{b}_{\mathfrak{c}}, \mathfrak{b}_{\mathfrak{c}}\}$ satisfying the uniform property such that \mathcal{M}_2 is isomorphic to the quotient algebra $\{\mathfrak{b}_{\mathfrak{c}}, \mathfrak{b}_{\mathfrak{c}}\}/\mathcal{K}$ and moreover \mathcal{L} is isomorphic to $\mathcal{L}(\mathfrak{b}_{\mathfrak{c}}, \mathcal{K})$. If the root system R is reduced, our method suggests another approach to characterize Lie algebras graded by R compared with what is offered by E. Neher [11].

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2. PRELIMINARIES

Throughout this work, \mathbb{N} denotes the set of nonnegative integers and \mathbb{F} is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over \mathbb{F} . We denote the dual space of a vector space V by V^* . For a linear transformation T on a vector space V , if the trace of T is defined, we denote it by $tr(T)$. Also for a nonempty set S , by id_S (or id if there is no confusion), we mean the identity map on S and by $|S|$, we mean the cardinality of S . Finally for an index set I , by a conventional notation, we take $\bar{I} := \{\bar{i} \mid i \in I\}$ to be a disjoint copy of I and for each subset J of I , by \bar{J} , we mean the subset of \bar{I} corresponding to J .

2.1. Locally Finite Split Simple Lie Algebras. In this subsection, we recall the structure of infinite dimensional locally finite split simple Lie algebras from [10] and state some facts which play key roles in this work. Let us start with the following definition.

Definition 2.1. Let \mathcal{H} be a Lie algebra. We say an \mathcal{H} -module \mathcal{M} has a *weight space decomposition with respect to \mathcal{H}* , if

$$\mathcal{M} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{M}_\alpha \text{ where } \mathcal{M}_\alpha := \{x \in \mathcal{M} \mid h \cdot x = \alpha(h)x; \forall h \in \mathcal{H}\}$$

for all $\alpha \in \mathcal{H}^*$. The set $R := \{\alpha \in \mathcal{H}^* \mid \mathcal{M}_\alpha \neq \{0\}\}$ is called the *set of weights* of \mathcal{M} (with respect to \mathcal{H}). For $\alpha \in R$, \mathcal{M}_α is called a *weight space*, and any element of \mathcal{M}_α is called a *weight vector* of *weight* α . If a Lie algebra \mathcal{L} has a weight space decomposition with respect to a nontrivial subalgebra H of \mathcal{L} via the adjoint representation, H is called a *split toral subalgebra*. The set of weights of \mathcal{L} is called the *root system* of \mathcal{L} with respect to H , and the corresponding weight spaces are called *root spaces* of \mathcal{L} . A Lie algebra \mathcal{L} is called *split* if it contains a *splitting Cartan subalgebra*, that is a split toral subalgebra H of \mathcal{L} with $\mathcal{L}_0 = H$.

The root system of a locally finite split simple Lie algebra with respect to a splitting Cartan subalgebra is a reduced irreducible locally finite root system in the following sense (see [8] and [10]):

Definition 2.2. [9] Let \mathcal{U} be a nontrivial vector space and R be a subset of \mathcal{U} , R is said to be a *locally finite root system in \mathcal{U}* of *rank* $\dim(\mathcal{U})$ if the following are satisfied:

- (i) R is locally finite, contains zero and spans \mathcal{U} .
- (ii) For every $\alpha \in R^\times := R \setminus \{0\}$, there exists $\check{\alpha} \in \mathcal{U}^*$ such that $\check{\alpha}(\alpha) = 2$ and $s_\alpha(\beta) \in R$ for $\alpha, \beta \in R$ where $s_\alpha : \mathcal{U} \rightarrow \mathcal{U}$ maps $u \in \mathcal{U}$ to $u - \check{\alpha}(u)\alpha$. We set by convention $\check{0}$ to be zero.
- (iii) $\check{\alpha}(\beta) \in \mathbb{Z}$, for $\alpha, \beta \in R$.

Set $R_{sdiv} := (R \setminus \{\alpha \in R \mid 2\alpha \in R\}) \cup \{0\}$ and call it the *semi-divisible subsystem* of R . The root system R is called *reduced* if $R = R_{sdiv}$.

Suppose that R is a locally finite root system. A nonempty subset S of R is said to be a *subsystem* of R if S contains zero and $s_\alpha(\beta) \in S$ for $\alpha, \beta \in S \setminus \{0\}$. A subsystem S of R is called *full* if $\text{span}_{\mathbb{F}} S \cap R = S$. Following [9, §2.6], we say two nonzero roots α, β are *connected* if there exist finitely many roots $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n = \beta$ such that $\check{\alpha}_{i+1}(\alpha_i) \neq 0, 1 \leq i \leq n-1$. Connectedness defines an equivalence relation on R^\times and so R^\times is the disjoint union of its equivalence classes called *connected components* of R . A nonempty subset X of R is called *irreducible*, if each two nonzero elements $x, y \in X$ are connected and it is called *closed* if $(X + X) \cap R \subseteq X$. It is easy to see that if X is a connected component of a locally finite root system R , then $X \cup \{0\}$ is a closed subsystem of R . For the locally finite root system R , take $\{R_\lambda \mid \lambda \in \Gamma\}$ to be the class of all finite subsystems of R , and say $\lambda \preccurlyeq \mu$ ($\lambda, \mu \in \Gamma$) if R_λ is a subsystem of R_μ , then (Γ, \preccurlyeq) is a directed set and R is

the direct union of $\{R_\lambda \mid \lambda \in \Gamma\}$. Furthermore, if R is irreducible, it is the direct union of its irreducible finite subsystems.

Two locally finite root systems (R, \mathcal{U}) and (S, \mathcal{V}) are said to be isomorphic if there is a linear transformation $f : \mathcal{U} \rightarrow \mathcal{V}$ such that $f(R) = S$.

Suppose that I is a nonempty index set and $\mathcal{U} := \bigoplus_{i \in I} \mathbb{F} \epsilon_i$ is the free \mathbb{F} -module over the set I . Define the form

$$\begin{aligned} (\cdot, \cdot) : \mathcal{U} \times \mathcal{U} &\longrightarrow \mathbb{F} \\ (\epsilon_i, \epsilon_j) &= \delta_{i,j}, \text{ for } i, j \in I \end{aligned}$$

and set

$$\begin{aligned} \dot{A}_I &:= \{\epsilon_i - \epsilon_j \mid i, j \in I\}, \\ D_I &:= \dot{A}_I \cup \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in I, i \neq j\}, \\ B_I &:= D_I \cup \{\pm\epsilon_i \mid i \in I\}, \\ C_I &:= D_I \cup \{\pm 2\epsilon_i \mid i \in I\}, \\ BC_I &:= B_I \cup C_I. \end{aligned} \tag{2.3}$$

One can see that these are irreducible locally finite root systems in their \mathbb{F} -span's which we refer to as *type A, B, C, D* and *BC* respectively. Moreover every irreducible locally finite root system of infinite rank is isomorphic to one of these root systems (see [9, §4.14 §8]). Now we suppose R is an irreducible locally finite root system as above and note that $(\alpha, \alpha) \in \mathbb{N}$ for all $\alpha \in R$. This allows us to define

$$\begin{aligned} R_{sh} &:= \{\alpha \in R^\times \mid (\alpha, \alpha) \leq (\beta, \beta); \text{ for all } \beta \in R\}, \\ R_{ex} &:= R \cap 2R_{sh}, \\ R_{lg} &:= R^\times \setminus (R_{sh} \cup R_{ex}). \end{aligned}$$

The elements of R_{sh} (resp. R_{lg}, R_{ex}) are called *short roots* (resp. *long roots, extra-long roots*) of R .

A locally finite split simple Lie algebra is said to be of type *A, B, C* or *D* if its corresponding root system with respect to a splitting Cartan subalgebra is of type *A, B, C* or *D* respectively. In what follows, we recall from [10] the classification of infinite dimensional locally finite split simple Lie algebras. Suppose that J is an index set and $\mathcal{V} = \mathcal{V}_J$ is a vector space with a fixed basis $\{v_j \mid j \in J\}$. One knows that $\mathfrak{gl}(\mathcal{V}) := \text{End}(\mathcal{V})$ together with

$$[\cdot, \cdot] : \mathfrak{gl}(\mathcal{V}) \times \mathfrak{gl}(\mathcal{V}) \longrightarrow \mathfrak{gl}(\mathcal{V}); (X, Y) \mapsto XY - YX; \quad X, Y \in \mathfrak{gl}(\mathcal{V})$$

is a Lie algebra. Now for $j, k \in J$, define

$$e_{j,k} : \mathcal{V} \longrightarrow \mathcal{V}; \quad v_i \mapsto \delta_{k,i} v_j, \quad (i \in J), \tag{2.4}$$

then $\mathfrak{gl}(J) := \text{span}_{\mathbb{F}}\{e_{j,k} \mid j, k \in J\}$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{V})$.

Lemma 2.5 (Classical Lie algebras of type *A*). *Suppose that I is a non-empty index set of cardinality greater than 1, I_0 is a fixed subset of I with $|I_0| > 1$ and \mathcal{V} is a vector space with a basis $\{v_i \mid i \in I\}$. Take Λ to be an index set containing 0 and $\{I_\lambda \mid \lambda \in \Lambda\}$ to be the class of all finite subsets of I containing I_0 . Set*

$$\mathcal{G} := \mathfrak{sl}(I) := \{\phi \in \mathfrak{gl}(I) \mid \text{tr}(\phi) = 0\},$$

and for $\lambda \in \Lambda$, take

$$\mathcal{G}_{I_\lambda} := \mathcal{G}^\lambda := \mathcal{G} \cap \text{span}\{e_{r,s} \mid r, s \in I_\lambda\}.$$

Then $\mathfrak{sl}(I)$ is a locally finite split simple Lie subalgebra of $\mathfrak{gl}(I)$ with splitting Cartan subalgebra $\mathcal{H} := \text{span}\{e_{i,i} - e_{j,j} \mid i, j \in I\}$ and corresponding root system isomorphic to A_I . Moreover for $i, j \in I$ with $i \neq j$, we have

$$\mathcal{G}_{\epsilon_i - \epsilon_j} = \mathbb{F}e_{i,j}.$$

Also for each $\lambda \in \Lambda$, \mathcal{G}^λ is a finite dimensional split simple Lie subalgebra of \mathcal{G} with splitting Cartan subalgebra $\mathcal{H}^\lambda := \mathcal{H} \cap \mathcal{G}^\lambda$ and \mathcal{G} is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$.

In the following lemma, we see that locally finite split simple Lie algebras of type B can be described in terms of derivations of *Clifford Jordan algebras* which are defined as following:

Definition 2.6 ([13]). Suppose that A is a unital commutative associative algebra over \mathbb{F} and \mathcal{W} is a unitary A -module. Suppose that $g : \mathcal{W} \times \mathcal{W} \rightarrow A$ is a symmetric A -bilinear form and set $\mathcal{J} = \mathcal{J}(g, \mathcal{W}) := A \oplus \mathcal{W}$. The vector space \mathcal{J} together with the following multiplication

$$(a_1 + w_1)(a_2 + w_2) = a_1a_2 + g(w_1, w_2) + a_1w_2 + a_2w_1$$

for $a_1, a_2 \in A$ and $w_1, w_2 \in \mathcal{W}$ is a Jordan algebra called a *Clifford Jordan algebra*. For $a, b \in \mathcal{J}$, define $D_{a,b} := -[\mathbf{L}_a, \mathbf{L}_b] := \mathbf{L}_b\mathbf{L}_a - \mathbf{L}_a\mathbf{L}_b$ where $\mathbf{L}_a, \mathbf{L}_b$ are left multiplications by a and b respectively. For a subspace V of \mathcal{J} , set $D_{V,V}$ to be the subspace of endomorphisms of \mathcal{J} spanned by $D_{a,b}$ for $a, b \in V$. One can see that for $w_1, w_2 \in \mathcal{W}$, D_{w_1, w_2} can be identified with $D_{w_1, w_2}|_{\mathcal{W}}$. This allows us to consider $D_{\mathcal{W}, \mathcal{W}}$ as a subalgebra of $\mathfrak{gl}(\mathcal{W})$.

Lemma 2.7 (Classical Lie algebras of type B). *Suppose that I is a non-empty index set. Take $J := \{0\} \uplus I \uplus \bar{I}$ and consider the vector space $\mathcal{V} := \mathcal{V}_J$ as before. Define the bilinear form (\cdot, \cdot) on \mathcal{V} by*

$$(2.8) \quad \begin{aligned} (v_j, v_{\bar{k}}) &= (v_{\bar{k}}, v_j) = 2\delta_{j,k}, \quad (v_0, v_0) = 2, \\ (v_j, v_k) &= (v_j, v_0) = (v_0, v_j) = (v_0, v_{\bar{j}}) = (v_{\bar{j}}, v_0) = (v_{\bar{j}}, v_{\bar{k}}) = 0; \quad j, k \in I, \end{aligned}$$

and set

$$\mathcal{G} := \mathfrak{o}_B(I) := \{\phi \in \mathfrak{gl}(J) \mid (\phi(v), w) = -(v, \phi(w)), \text{ for all } v, w \in \mathcal{V}\}.$$

Then we have the following:

(i) \mathcal{G} is a locally finite split simple Lie subalgebra of $\mathfrak{gl}(J)$ with splitting Cartan subalgebra $\mathcal{H} := \text{span}_{\mathbb{F}}\{h_i := e_{i,i} - e_{\bar{i},\bar{i}} \mid i \in I\}$ and corresponding root system isomorphic to B_I . Moreover for $i, j \in J$ with $i \neq j$, we have

$$\begin{aligned} \mathcal{G}_{\epsilon_i - \epsilon_j} &= \mathbb{F}(e_{i,j} - e_{\bar{j},\bar{i}}), \quad \mathcal{G}_{\epsilon_i + \epsilon_j} = \mathbb{F}(e_{i,\bar{j}} - e_{\bar{j},i}), \quad \mathcal{G}_{-\epsilon_i - \epsilon_j} = \mathbb{F}(e_{\bar{i},j} - e_{j,\bar{i}}) \\ \mathcal{G}_{\epsilon_i} &= \mathbb{F}(e_{i,0} - e_{0,\bar{i}}), \quad \mathcal{G}_{-\epsilon_i} = \mathbb{F}(e_{\bar{i},0} - e_{0,i}). \end{aligned}$$

(ii) For the Clifford Jordan algebra $\mathcal{J}((\cdot, \cdot), \mathcal{V})$, we have $\mathcal{G} = D_{\mathcal{V}, \mathcal{V}}$.

(iii) For a fixed subset I_0 of I , take Λ to be an index set containing 0 such that $\{I_\lambda \mid \lambda \in \Lambda\}$ is the class of all finite subsets of I containing I_0 . For each $\lambda \in \Lambda$, set

$$(2.9) \quad \mathcal{G}_{I_\lambda} := \mathcal{G}^\lambda := \mathcal{G} \cap \text{span}\{e_{r,s} \mid r, s \in \{0\} \cup I_\lambda \cup \bar{I}_\lambda\}.$$

Then \mathcal{G}^λ ($\lambda \in \Lambda$) is a finite dimensional split simple Lie subalgebra of \mathcal{G} of type B , with splitting Cartan subalgebra $\mathcal{H}^\lambda := \mathcal{H} \cap \mathcal{G}^\lambda$ and \mathcal{G} is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$.

Lemma 2.10 (Classical Lie algebras of type D). *Suppose that I is a non-empty index set and I_0 is a fixed subset of I . Set $J := I \uplus \bar{I}$ and take $\{I_\lambda \mid \lambda \in \Lambda\}$, where Λ is an index set containing 0, to be the class of all finite subsets of I containing I_0 . Define the bilinear form (\cdot, \cdot) on $\mathcal{V} = \mathcal{V}_J$ by*

$$(2.11) \quad (v_j, v_{\bar{k}}) = (v_{\bar{k}}, v_j) = 2\delta_{j,k}, \quad (v_j, v_k) = (v_{\bar{j}}, v_{\bar{k}}) = 0; \quad (j, k \in I),$$

and set

$$\begin{aligned} \mathcal{G} &:= \mathfrak{o}_D(I) := \{\phi \in \mathfrak{gl}(J) \mid (\phi(v), w) = -(v, \phi(w)), \text{ for all } v, w \in \mathcal{V}\}, \\ \mathcal{H} &:= \text{span}_{\mathbb{F}}\{h_i := e_{i,i} - e_{\bar{i},\bar{i}} \mid i \in I\}. \end{aligned}$$

Also for $\lambda \in \Lambda$, take

$$\mathcal{G}_{I_\lambda} := \mathcal{G}^\lambda := \mathcal{G} \cap \text{span}\{e_{r,s} \mid r, s \in I_\lambda \cup \bar{I}_\lambda\}.$$

Then \mathcal{G} is a locally finite split simple Lie subalgebra of $\mathfrak{gl}(J)$ with splitting Cartan subalgebra \mathcal{H} and corresponding root system isomorphic to D_I . Moreover for $i, j \in J$ with $i \neq j$, we have

$$\mathcal{G}_{\epsilon_i - \epsilon_j} = \mathbb{F}(e_{i,j} - e_{\bar{j},\bar{i}}), \quad \mathcal{G}_{\epsilon_i + \epsilon_j} = \mathbb{F}(e_{i,\bar{j}} - e_{j,\bar{i}}), \quad \mathcal{G}_{-\epsilon_i - \epsilon_j} = \mathbb{F}(e_{\bar{i},j} - e_{\bar{j},i}).$$

Also for each $\lambda \in \Lambda$, \mathcal{G}^λ is a finite dimensional split simple Lie subalgebra of \mathcal{G} , of type D , with splitting Cartan subalgebra $\mathcal{H}^\lambda := \mathcal{H} \cap \mathcal{G}^\lambda$, and \mathcal{G} is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$.

Lemma 2.12 (Classical Lie algebras of type C). *Suppose that I is a non-empty index set and $J := I \uplus \bar{I}$. Consider the bilinear form (\cdot, \cdot) on $\mathcal{V} = \mathcal{V}_J$ defined by*

$$(2.13) \quad (v_j, v_{\bar{k}}) = -(v_{\bar{k}}, v_j) = 2\delta_{j,k}, \quad (v_j, v_k) = 0, \quad (v_{\bar{j}}, v_{\bar{k}}) = 0, \quad (j, k \in I),$$

and set

$$\mathcal{G} := \mathfrak{sp}(I) := \{\phi \in \mathfrak{gl}(J) \mid (\phi(v), w) = -(v, \phi(w)), \text{ for all } v, w \in \mathcal{V}\}.$$

Also for a fixed subset I_0 of I , take $\{I_\lambda \mid \lambda \in \Lambda\}$ to be the class of all finite subsets of I containing I_0 , in which Λ is an index set containing 0, and for each $\lambda \in \Lambda$, set

$$(2.14) \quad \mathcal{G}_{I_\lambda} := \mathcal{G}^\lambda := \mathcal{G} \cap \text{span}\{e_{r,s} \mid r, s \in I_\lambda \cup \bar{I}_\lambda\}.$$

Then \mathcal{G} is a locally finite split simple Lie subalgebra of $\mathfrak{gl}(J)$ with splitting Cartan subalgebra $\mathcal{H} := \text{span}_{\mathbb{F}}\{h_i := e_{i,i} - e_{\bar{i},\bar{i}} \mid i \in I\}$. Moreover for $i, j \in I$ with $i \neq j$, we have

$$\begin{aligned} \mathcal{G}_{\epsilon_i - \epsilon_j} &= \mathbb{F}(e_{i,j} - e_{\bar{j},\bar{i}}), \quad \mathcal{G}_{\epsilon_i + \epsilon_j} = \mathbb{F}(e_{i,\bar{j}} + e_{j,\bar{i}}), \quad \mathcal{G}_{-\epsilon_i - \epsilon_j} = \mathbb{F}(e_{\bar{i},j} + e_{\bar{j},i}) \\ \mathcal{G}_{2\epsilon_i} &= \mathbb{F}e_{i,\bar{i}}, \quad \mathcal{G}_{-2\epsilon_i} = \mathbb{F}e_{\bar{i},i}. \end{aligned}$$

Also for $\lambda \in \Lambda$, \mathcal{G}^λ is a finite dimensional split simple Lie subalgebra of type C , with splitting Cartan subalgebra $\mathcal{H}^\lambda := \mathcal{H} \cap \mathcal{G}^\lambda$, and \mathcal{G} is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$.

Proposition 2.15. [10, Theorem VI.7] *Suppose that I is an infinite index set, then $\mathfrak{o}_B(I)$ is isomorphic to $\mathfrak{o}_D(I)$. Moreover if \mathcal{G} is an infinite dimensional locally finite split simple Lie algebra, then \mathcal{G} is isomorphic to exactly one of the Lie algebras $\mathfrak{sl}(I)$, $\mathfrak{o}_B(I)$ or $\mathfrak{sp}(I)$.*

Lemma 2.16. *Suppose that R is an irreducible locally finite root system and S is an irreducible closed subsystem of R . Suppose \mathcal{G} is a locally finite split simple Lie algebra with a splitting Cartan subalgebra \mathcal{H} and the root system R_{sdiv} . Set $\mathfrak{g} := \sum_{\alpha \in S_{sdiv}^\times} \mathcal{G}_\alpha \oplus \sum_{\alpha \in S_{sdiv}^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ and $\mathfrak{h} := \mathcal{H} \cap \mathfrak{g}$, then the restriction of*

$$\pi : \mathcal{H}^* \longrightarrow \mathfrak{h}^*; \quad f \mapsto f|_{\mathfrak{h}}, \quad f \in \mathcal{H}^*$$

to S is injective. Identify $\alpha \in S$ with $\pi(\alpha)$ via π , then \mathfrak{g} is a locally finite split simple Lie subalgebra of \mathcal{G} with splitting Cartan subalgebra \mathfrak{h} and corresponding root system S_{sdiv} .

Proof. We first claim that

$$(2.17) \quad \begin{aligned} &\text{if } \alpha, \beta \in S \text{ and } \alpha - \beta \notin R, \text{ then there is} \\ &h \in \mathfrak{h} \text{ such that } \alpha(h) > 0 \text{ and } \beta(h) \leq 0. \end{aligned}$$

To prove this, we note that since $\alpha - \beta \notin R$, we have $\alpha \neq 0$ and $\beta \neq 0$. Moreover, it follows from the theory of locally finite root systems that $\beta - 2\alpha \notin R$ and $\alpha - 2\beta \notin R$, also if $2\alpha \in R$ or $2\beta \in R$, then $2\alpha - 2\beta \notin R$. Therefore setting

$$\alpha' := \begin{cases} \alpha & \text{if } 2\alpha \notin R \\ 2\alpha & \text{if } 2\alpha \in R, \end{cases} \quad \text{and} \quad \beta' := \begin{cases} \beta & \text{if } 2\beta \notin R \\ 2\beta & \text{if } 2\beta \in R, \end{cases}$$

we have $\alpha', \beta' \in S_{sdiv}^\times$ and $\alpha' - \beta' \notin R$. Next we fix $e \in \mathcal{G}_{\alpha'}$ and $f \in \mathcal{G}_{-\alpha'}$ such that $(e, h := [e, f], f)$ is an \mathfrak{sl}_2 -triple. Since $\alpha' - \beta' \notin R_{sdiv}$, one knows from \mathfrak{sl}_2 -module theory that $\beta'(h) \leq 0$ while $\alpha'(h) = 2 > 0$. Therefore $h \in [\mathcal{G}_{\alpha'}, \mathcal{G}_{-\alpha'}] \subseteq \mathfrak{h}$, $\alpha(h) > 0$ and $\beta(h) \leq 0$. This completes the proof of the claim. Now suppose $\alpha, \beta \in S$ with $\pi(\alpha) = \pi(\beta)$. We must show $\alpha = \beta$. We prove this through the following three cases:

Case 1. $\alpha, \beta \in S_{sdiv}$: If $\gamma := \alpha - \beta \in R^\times$, then since S is a closed subsystem of R and S_{sdiv} is a closed subsystem of S , we get $\gamma \in S_{sdiv}^\times$. Thus there is $t \in [\mathcal{G}_\gamma, \mathcal{G}_{-\gamma}] \subseteq \mathfrak{h}$ with $\gamma(t) = 2$, so $(\alpha - \beta)(t) = 2$ which contradicts the fact that $\alpha|_{\mathfrak{h}} = \beta|_{\mathfrak{h}}$. Therefore $\alpha - \beta \notin R^\times$. Now if $\alpha - \beta \neq 0$, then

$\alpha - \beta \notin R$ and so using (2.17), one finds $h \in \mathfrak{h}$ with $\alpha(h) > 0$ and $\beta(h) \leq 0$. This is again a contradiction. Therefore $\alpha = \beta$.

Case 2. $\alpha, \beta \notin S_{sdiv}$: In this case $2\alpha, 2\beta \in S_{sdiv}$ and so by Case 1, $2\alpha = 2\beta$ which in turn implies that $\alpha = \beta$.

Case 3. $\alpha \in S_{sdiv}, \beta \notin S_{sdiv}$: If $\alpha - \beta \notin R$, then by (2.17), there is $h \in \mathfrak{h}$ such that $\alpha(h) > 0$ and $\beta(h) \leq 0$ which contradicts the fact that $\pi(\alpha) = \pi(\beta)$. Also if $\alpha = 2\beta$, then since $\alpha \in S_{sdiv}^\times$, there is $h \in [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] \subseteq \mathfrak{h}$ with $\alpha(h) = 2$. Thus $\alpha(h) \neq \beta(h)$ which is again a contradiction. Therefore $\alpha - \beta \in R$ and $\alpha \neq 2\beta$. Now if $\alpha - \beta \neq 0$, we get that $\alpha - \beta \in R_{sh}$, $\alpha \in R_{lg}$, $\beta \in R_{sh}$, $\gamma := \alpha - 2\beta \in R_{lg}$, $\alpha + \gamma, \alpha - \gamma \in R_{ex}$ and $\alpha + 2\gamma, \alpha - 2\gamma \notin R$. Now since $\gamma \in S_{sdiv}^\times$, there is $h \in [\mathcal{G}_\gamma, \mathcal{G}_{-\gamma}] \subseteq \mathfrak{h}$ with $\gamma(h) = 2$. Also since $\alpha + 2\gamma, \alpha - 2\gamma \notin R$, one concludes from \mathfrak{sl}_2 -module theory that $\alpha(h) = 0$. So we have $\beta(h) = \alpha(h) = 0$. But this gives that $2 = \gamma(h) = (\alpha - 2\beta)(h) = 0$, a contradiction. Thus we have $\alpha = \beta$. This completes the proof of the first assertion.

For the last assertion, we note that S is a closed subsystem of R and S_{sdiv} is a closed subsystem of S , so it is easily seen that \mathfrak{g} is a subalgebra of \mathcal{G} . Now this together with the fact that $\pi|_S$ is injective completes the proof. \square

Definition 2.18. Take \mathcal{G} , Λ , \mathcal{G}^λ , and \mathcal{H}^λ ($\lambda \in \Lambda$) to be as in one of Lemmas 2.5, 2.7, 2.10 and 2.12. For $\lambda, \mu \in \Lambda$, we say $\lambda \preccurlyeq \mu$ if \mathcal{G}^λ is a subalgebra of \mathcal{G}^μ . Let χ be a representation of \mathcal{G} in a vector space \mathcal{M} . We say \mathcal{M} is a *direct limit \mathcal{G} -module with directed system $\{\mathcal{M}^\lambda \mid \lambda \in \Lambda\}$* if

- for $\lambda, \mu \in \Lambda$ with $\lambda \preccurlyeq \mu$, $\mathcal{M}^\lambda \subseteq \mathcal{M}^\mu \subseteq \mathcal{M}$ and as a vector space, \mathcal{M} is the direct union of $\{\mathcal{M}^\lambda \mid \lambda \in \Lambda\}$,
- for $\lambda \in \Lambda$, \mathcal{M}^λ is finite dimensional and for all $x \in \mathcal{G}^\lambda$, \mathcal{M}^λ is invariant under $\chi(x)$,
- for $\lambda \in \Lambda$, $\chi|_{\mathcal{G}^\lambda}$ defines a nontrivial finite dimensional irreducible \mathcal{G}^λ -module in \mathcal{M}^λ having a weight space decomposition with respect to \mathcal{H}^λ whose weight spaces corresponding to nonzero weights are one dimensional,
- for $\lambda, \mu \in \Lambda$ with $\lambda \preccurlyeq \mu$, the set of weights of \mathcal{G}^λ -module \mathcal{M}^λ is contained in the set of weights of \mathcal{G}^μ -module \mathcal{M}^μ restricted to \mathcal{H}^λ and $(\mathcal{M}^\lambda)_{p|_{\mathcal{H}^\lambda}} = (\mathcal{M}^\mu)_p$ for each weight p of \mathcal{M}^μ for which $p|_{\mathcal{H}^\lambda}$ is a nonzero weight of \mathcal{M}^λ .

Using standard techniques, one can verify the following propositions:

Proposition 2.19. Consider \mathcal{G} , Λ , \mathcal{G}^λ and \mathcal{H}^λ ($\lambda \in \Lambda$) as in Definition 2.18. Suppose that \mathcal{M} is a direct limit \mathcal{G} -module with directed system $\{\mathcal{M}^\lambda \mid \lambda \in \Lambda\}$. Then we have the followings:

- \mathcal{M} is an irreducible \mathcal{G} -module.
- If \mathcal{W} is another direct limit \mathcal{G} -module with directed system $\{\mathcal{W}^\lambda \mid \lambda \in \Lambda\}$, and for each $\lambda \in \Lambda$, \mathcal{G}^λ -modules \mathcal{M}^λ and \mathcal{W}^λ are isomorphic, then as two \mathcal{G} -modules, \mathcal{M} and \mathcal{W} are isomorphic.

Proposition 2.20. *Suppose that I is a nonempty index set.*

(a) *Take $\mathcal{G} := \mathfrak{o}_B(I)$ and use the same notations as in Lemma 2.7. Define*

$$\pi : \mathcal{G} \longrightarrow \text{End}(\mathcal{V}); \pi(\phi)(v) = \phi(v); \phi \in \mathcal{G}, v \in \mathcal{V},$$

then

(i) *π is an irreducible representation of \mathcal{G} in \mathcal{V} equipped with a weight space decomposition with respect to \mathcal{H} whose set of weights is $\{0, \pm\epsilon_i \mid i \in I\}$ with $\mathcal{V}_0 = \mathbb{F}v_0$, $\mathcal{V}_{\epsilon_i} = \mathbb{F}v_i$ and $\mathcal{V}_{-\epsilon_i} = \mathbb{F}v_{\bar{i}}$ for $i \in I$,*

(ii) *for each $\lambda \in \Lambda$, set*

$$(2.21) \quad \mathcal{V}_{I_\lambda} := \mathcal{V}^\lambda := \text{span}_{\mathbb{F}}\{v_r \mid r \in \{0\} \cup I_\lambda \cup \bar{I}_\lambda\},$$

then \mathcal{V} is a direct limit \mathcal{G} -module with directed system $\{\mathcal{V}^\lambda \mid \lambda \in \Lambda\}$.

(b) *Use the same notations as in Lemma 2.12 and take $\mathcal{G} := \mathfrak{sp}(I)$.*

(i) *Define*

$$\pi_1 : \mathcal{G} \longrightarrow \text{End}(\mathcal{V}); \pi_1(\phi)(v) := \phi(v); \phi \in \mathcal{G}, v \in \mathcal{V}.$$

Then π_1 is an irreducible representation of \mathcal{G} in \mathcal{V} equipped with a weight space decomposition with respect to \mathcal{H} whose set of weights is $\{\pm\epsilon_i \mid i \in I\}$ with $\mathcal{V}_{\epsilon_i} = \mathbb{F}v_i$ and $\mathcal{V}_{-\epsilon_i} = \mathbb{F}v_{\bar{i}}$ for $i \in I$. Also for $J := I \cup \bar{I}$ and

$$(2.22) \quad \mathcal{S} := \{\phi \in \mathfrak{gl}(J) \mid \text{tr}(\phi) = 0, (\phi(v), w) = (v, \phi(w)), \text{ for all } v, w \in \mathcal{V}\},$$

$$\pi_2 : \mathcal{G} \longrightarrow \text{End}(\mathcal{S}); \pi_2(X)(Y) := [X, Y]; X \in \mathcal{G}, Y \in \mathcal{S}$$

is an irreducible representation of \mathcal{G} in \mathcal{S} equipped with a weight space decomposition with respect to \mathcal{H} whose set of weights is $\{0, \pm(\epsilon_i \pm \epsilon_j) \mid i, j \in I, i \neq j\}$ with $\mathcal{S}_0 = \text{span}_{\mathbb{F}}\{e_{r,r} + e_{\bar{r},\bar{r}} - \frac{1}{|I_\lambda|} \sum_{i \in I_\lambda} (e_{i,i} + e_{\bar{i},\bar{i}}) \mid \lambda \in \Lambda, r \in I_\lambda\}$, $\mathcal{S}_{\epsilon_i + \epsilon_j} = \mathbb{F}(e_{i,\bar{j}} - e_{j,\bar{i}})$, $\mathcal{S}_{-\epsilon_i - \epsilon_j} = \mathbb{F}(e_{\bar{i},j} - e_{\bar{j},i})$ and $\mathcal{S}_{\epsilon_i - \epsilon_j} = \mathbb{F}(e_{i,j} + e_{j,\bar{i}})$ ($i, j \in I, i \neq j$).

(ii) *For $\lambda \in \Lambda$, set*

$$(2.23) \quad \begin{aligned} \mathcal{V}_{I_\lambda} &:= \mathcal{V}^\lambda := \text{span}_{\mathbb{F}}\{v_r \mid r \in I_\lambda \cup \bar{I}_\lambda\}, \\ \mathcal{S}_{I_\lambda} &:= \mathcal{S}^\lambda := \mathcal{S} \cap \text{span}_{\mathbb{F}}\{e_{r,s} \mid r, s \in I_\lambda \cup \bar{I}_\lambda\}. \end{aligned}$$

Then \mathcal{V} is a direct limit \mathcal{G} -module with directed system $\{\mathcal{V}^\lambda \mid \lambda \in \Lambda\}$ and \mathcal{S} is a direct limit \mathcal{G} -module with directed system $\{\mathcal{S}^\lambda \mid \lambda \in \Lambda\}$.

2.2. Finite Dimensional Case. In this subsection, we state a proposition on representation theory of finite dimensional split simple Lie algebras. This proposition is an essential tool for the proof of our results in the next section. We start with an elementary but important fact about finite dimensional representations of a finite dimensional split semisimple Lie algebra.

Lemma 2.24. *Suppose that \mathcal{G} is a finite dimensional split semisimple Lie algebra with a splitting Cartan subalgebra \mathcal{H} and the root system R . Let \mathcal{V} be a finite dimensional \mathcal{G} -module equipped with a weight space decomposition with respect to \mathcal{H} . Take Π to be the set of weights of \mathcal{V} . If $\alpha \in R^\times$ and $\lambda \in \Pi$ are such that $\alpha + \lambda \in \Pi$, then $\mathcal{G}_\alpha \cdot \mathcal{V}_\lambda \neq \{0\}$. In particular if $\mathcal{V}_{\lambda+\alpha}$ is one dimensional, then $\mathcal{G}_\alpha \cdot \mathcal{V}_\lambda = \mathcal{V}_{\lambda+\alpha}$.*

Proof. Take $e \in \mathcal{G}_\alpha$ and $f \in \mathcal{G}_{-\alpha}$ to be such that $(e, h := [e, f], f)$ is an \mathfrak{sl}_2 -triple and define $\mathfrak{s} := \text{span}_{\mathbb{F}}\{e, h, f\}$. Set $\mathcal{W} := \sum_{k=-\infty}^{\infty} \mathcal{V}_{\lambda+k\alpha}$, then \mathcal{W} is a finite dimensional \mathfrak{s} -module and so by Weyl theorem, it is decomposed into finite dimensional irreducible \mathfrak{s} -modules, say $\mathcal{W} = \oplus_{i=1}^n \mathcal{W}_i$ where n is a positive integer and \mathcal{W}_i , $1 \leq i \leq n$, is a finite dimensional irreducible \mathfrak{s} -module. We note that the set of weights of \mathcal{W} with respect to $\mathbb{F}h$ is $\Pi' = \{\lambda(h) + 2k \mid k \in \mathbb{Z} \text{ and } \lambda + k\alpha \in \Pi\}$ and that for $k \in \mathbb{Z}$ with $\lambda + k\alpha \in \Pi$, $\mathcal{W}_{\lambda(h)+2k} = \mathcal{V}_{\lambda+k\alpha}$. Now as $\lambda, \lambda + \alpha \in \Pi$, we have $\lambda(h), \lambda(h) + 2 \in \Pi'$ and so by \mathfrak{sl}_2 -module theory, there is $1 \leq i \leq n$ such that $\lambda(h), \lambda(h) + 2$ are weights for \mathcal{W}_i . Now again using \mathfrak{sl}_2 -module theory, we get that

$$0 \neq e \cdot (\mathcal{W}_i)_{\lambda(h)} \subseteq e \cdot \mathcal{W}_{\lambda(h)} = e \cdot \mathcal{V}_\lambda \subseteq \mathcal{G}_\alpha \cdot \mathcal{V}_\lambda$$

showing that $\mathcal{G}_\alpha \cdot \mathcal{V}_\lambda \neq \{0\}$. The last statement is derived simply from the first assertion. \square

Lemma 2.25. Suppose that $\{e_i, f_i, h_i \mid 1 \leq i \leq n\}$ is a set of Chevalley generators for a finite dimensional split simple Lie algebra \mathcal{G} and \mathcal{V} is a \mathcal{G} -module equipped with a weight space decomposition with respect to the Cartan subalgebra $\mathcal{H} := \text{span}\{h_i \mid 1 \leq i \leq n\}$. Let v be a weight vector, m be a positive integer and $1 \leq i, j_1, \dots, j_m \leq n$. Let the set $\{k \in \{1, \dots, m\} \mid j_k = i\}$ be a nonempty set and $k_1 < \dots < k_p$ be such that $\{k \in \{1, \dots, m\} \mid j_k = i\} = \{k_1, \dots, k_p\}$. Then if $f_i \cdot v = 0$, we have

$$f_i \cdot e_{j_m} \cdots e_{j_1} \cdot v \in \sum_{t=1}^p \mathbb{F} e_{j_m} \cdots e_{j_{k_p}} \cdots \widehat{e_{j_{k_t}}} \cdots e_{j_{k_1}} \cdots e_{j_1} \cdot v,$$

in which " $\widehat{}$ " means omission.

Proof. Using Induction on p , we are done. \square

Proposition 2.26. Suppose R_1 is an irreducible finite root system and R_2 is an irreducible full subsystem of R_1 of rank greater than 1. Let \mathcal{G}_1 be a finite dimensional split simple Lie algebra with a splitting Cartan subalgebra \mathcal{H}_1 and corresponding root system $(R_1)_{\text{div}}$. Set $\mathcal{G}_2 := \sum_{\alpha \in (R_2)_{\text{div}}}^{\times} (\mathcal{G}_1)_\alpha \oplus \sum_{\alpha \in (R_2)_{\text{div}}}^{\times} [(\mathcal{G}_1)_\alpha, (\mathcal{G}_1)_{-\alpha}]$ and $\mathcal{H}_2 := \mathcal{H}_1 \cap \mathcal{G}_2$. For $i = 1, 2$, assume \mathcal{V}_i is a \mathcal{G}_i -module equipped with a weight space decomposition with respect to \mathcal{H}_i and take Λ_i ($i = 1, 2$) to be the set of weights of \mathcal{V}_i with respect to \mathcal{H}_i . Suppose that

- (i) R_1 and R_2 are of the same type $X \neq G_2, F_4, E_{6,7,8}$,
- (ii) $\Lambda_1 \subseteq R_1$ and $\Lambda_2 \subseteq \{\alpha|_{\mathcal{H}_2} \mid \alpha \in R_2\}$,
- (iii) $\mathcal{V}_2 \subseteq \mathcal{V}_1$ with $(\mathcal{V}_2)_{\alpha|_{\mathcal{H}_2}} \subseteq (\mathcal{V}_1)_\alpha$, for $\alpha \in \{\beta \in R_2 \mid \beta|_{\mathcal{H}_2} \in \Lambda_2\} \setminus \{0\}$.

Let \mathcal{W} be a nontrivial finite dimensional irreducible \mathcal{G}_2 -submodule of \mathcal{V}_2 and take \mathcal{U} to be the \mathcal{G}_1 -submodule of \mathcal{V}_1 generated by \mathcal{W} , then \mathcal{U} is a finite dimensional irreducible \mathcal{G}_1 -module equipped with a weight space decomposition with respect to \mathcal{H}_1 whose set of nonzero weights is $(R_1)_{\text{sh}}$, (resp. $(R_1)_{\text{div}}^{\times}$,

or $((R_1)_{sdiv})_{sh}$) if the set of nonzero weights of \mathcal{W} is the set of elements of $(R_2)_{sh}$ (resp. $(R_2)_{sdiv}^\times$, or $((R_2)_{sdiv})_{sh}$) restricted to \mathcal{H}_2 .

Proof. Take $n := n_1$ and $\ell := n_2$ to be the rank of R_1 and R_2 respectively. Using Lemma 2.16, we identify $\beta \in R_2$ with $\beta|_{\mathcal{H}_2}$. Also without loss of generality, we assume R_1, R_2 and bases Δ_1, Δ_2 for $(R_1)_{sdiv}$ and $(R_2)_{sdiv}$ respectively are as in the following tables:

Type	$R_k (k = 1, 2)$
A	$\{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n_k + 1\} \cup \{0\}, n_k \geq 2$
B	$\{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq n_k\} \cup \{0\}, n_k \geq 2$
C	$\{\pm 2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq n_k\} \cup \{0\}, n_k \geq 3$
D	$\{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq n_k\} \cup \{0\}, n_k \geq 4$
BC	$\{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i, j \leq n_k\}, n_k \geq 2$

Type	$\Delta_k (k = 1, 2)$
A	$\{\alpha_i := \epsilon_{i+1} - \epsilon_i \mid 1 \leq i \leq n_k\}$
B	$\{\alpha_1 := \epsilon_1, \alpha_i := \epsilon_i - \epsilon_{i-1} \mid 2 \leq i \leq n_k\}$
C	$\{\alpha_1 := 2\epsilon_1, \alpha_i := \epsilon_i - \epsilon_{i-1} \mid 2 \leq i \leq n_k\}$
D	$\{\alpha_1 := \epsilon_1 + \epsilon_2, \alpha_i := \epsilon_i - \epsilon_{i-1} \mid 2 \leq i \leq n_k\}$
BC	$\{\alpha_1 := 2\epsilon_1, \alpha_i := \epsilon_i - \epsilon_{i-1} \mid 2 \leq i \leq n_k\}$

Suppose that \mathcal{W} is of highest pair (v, α) with respect to Δ_2 . Since the set of weights of \mathcal{W} is permuted by the Weyl group of R_2 , one gets that $\alpha = \alpha_*^2$ for $*$ in $\{sh, lg, ex\}$ where for $i = 1, 2$, α_{sh}^i (resp. α_{lg}^i , or α_{ex}^i) denotes the highest short (resp. long, or extra long) root of R_i with respect to Δ_i . Next suppose that $\{e_i, f_i, h_i \mid 1 \leq i \leq n\}$ is a set of Chevalley generators for \mathcal{G}_1 with respect to Δ_1 , then $\{e_i, f_i, h_i \mid 1 \leq i \leq \ell\}$ is a set of Chevalley generators for \mathcal{G}_2 with respect to Δ_2 . Now as \mathcal{U} is a \mathcal{G}_1 -submodule of \mathcal{V}_1 generated by v , we have

$$(2.27) \quad \mathcal{U} = \sum_{t, s \in \mathbb{N}} \mathbb{F}(f_{i_t} \cdots f_{i_1} \cdot e_{j_s} \cdots e_{j_1} \cdot v)$$

where $i_1, \dots, i_t, j_1, \dots, j_s \in \{1, \dots, n\}$. This implies that \mathcal{U} is finite dimensional as Λ_1 is a subset of the finite root system R_1 . So there are a positive integer p and irreducible finite dimensional \mathcal{G}_1 -submodules \mathcal{U}_j ($1 \leq j \leq p$) of \mathcal{U} such that $\mathcal{U} = \bigoplus_{j=1}^p \mathcal{U}_j$. But we know that v generates the \mathcal{G}_1 -submodule \mathcal{U} , and that $v \in \mathcal{U} \cap (\mathcal{V}_2)_\alpha \subseteq \mathcal{U} \cap (\mathcal{V}_1)_\alpha = \mathcal{U}_\alpha = \bigoplus_{j=1}^p (\mathcal{U}_j)_\alpha$, so

$$(2.28) \quad \text{for any } 1 \leq j \leq p, \text{ there is a nonzero element } u_j \in (\mathcal{U}_j)_\alpha \text{ such that } v = \sum_{j=1}^p u_j.$$

This in particular implies that each \mathcal{U}_j ($1 \leq j \leq p$) is a nontrivial irreducible \mathcal{G}_1 -module. But we know that for $1 \leq j \leq p$, the set of weights of \mathcal{U}_j is a subset of R_1 , and that it is permuted by the Weyl group of R_1 , so the highest weight of \mathcal{U}_j is α_*^1 for $*$ = sh, lg, ex . Therefore using the finite dimensional

theory, one knows that
(2.29)

the weight spaces of \mathcal{U}_j ($1 \leq j \leq p$) with respect to \mathcal{H}_1 corresponding to nonzero weights are one dimensional.

Now we are ready to proceed with the proof in the following three steps:

- Step 1. \mathcal{U}_t ($1 \leq t \leq p$) is a finite dimensional irreducible \mathcal{G}_1 -module of highest weight α_*^1 if $\alpha = \alpha_*^2$ for $*$ = sh, lg, ex .
- Step 2. $\dim(\mathcal{U}_\alpha) = 1$,
- Step 3. $p = 1$.

Step 1: We use a case-by-case argument to prove the desired point. We note that there is nothing to show if R_1 is of type A or D , and continue as following:

Type B : One can see that in this case $\alpha_{sh}^1 = \epsilon_n$, $\alpha_{sh}^2 = \epsilon_\ell$, $\alpha_{lg}^1 = \epsilon_n + \epsilon_{n-1}$ and $\alpha_{lg}^2 = \epsilon_\ell + \epsilon_{\ell-1}$. We first assume $\alpha = \alpha_{sh}^2$ and show that the highest weight of \mathcal{U}_t is the highest short root of R_1 . For this, it is enough to show that no long root is a weight for \mathcal{U}_t . Suppose to the contrary that the set of weights of \mathcal{U}_t contains a long root or equivalently contains all long roots. Setting $\beta := \epsilon_{\ell-1}$, we get $\alpha + \beta$ is a long root of R_1 and so $\alpha + \beta$ is a weight for \mathcal{U}_t . Now fix $x \in (\mathcal{G}_2)_\beta = (\mathcal{G}_1)_\beta$ and note that $\alpha + \beta$ is a weight for \mathcal{U}_t ; applying Lemma 2.24 together with (2.28) and (2.29), we have $x \cdot u_t \neq 0$. This gives that $0 \neq \sum_{j=1}^p x \cdot u_j = x \cdot v \in (\mathcal{G}_2)_\beta \cdot \mathcal{W}_\alpha \subseteq \mathcal{W}_{\alpha+\beta}$ which is a contradiction as $\alpha + \beta$ is a long root and cannot be a weight for \mathcal{W} . Therefore \mathcal{U}_t has no long root as a weight and so we are done in the case $\alpha = \alpha_{sh}^2$. Next suppose $\alpha = \alpha_{lg}^2$ and note that by (2.28), u_t is a weight vector of \mathcal{U}_t of weight α . Since α is a long root, the set of weights of \mathcal{U}_t contains all long roots and so the highest long root is the highest weight of \mathcal{U}_t .

Type C : In this case, we have $\alpha_{sh}^1 = \epsilon_{n-1} + \epsilon_n$, $\alpha_{sh}^2 = \epsilon_{\ell-1} + \epsilon_\ell$, $\alpha_{lg}^1 = 2\epsilon_n$ and $\alpha_{lg}^2 = 2\epsilon_\ell$. Setting $\beta := \epsilon_{\ell-1} - \epsilon_\ell$, we get $\alpha_{sh}^2 + \beta \in (R_1)_{lg}$. Now using the same argument as in Type B, we are done.

Type BC : In this case, we have $\alpha_{sh}^1 = \epsilon_n$, $\alpha_{sh}^2 = \epsilon_\ell$, $\alpha_{lg}^1 = \epsilon_{n-1} + \epsilon_n$, $\alpha_{lg}^2 = \epsilon_{\ell-1} + \epsilon_\ell$, $\alpha_{ex}^1 = 2\epsilon_n$ and $\alpha_{ex}^2 = 2\epsilon_\ell$. One can also easily see that

$$\{\gamma + \beta \mid \gamma \in (R_1)_{sh}, \beta \in (R_1)_{lg} \cup (R_1)_{ex}\} \cap R_1 \subseteq (R_1)_{sh}.$$

This together with (2.27), (2.28) and the fact that $\Delta_1 \subseteq (R_1)_{ex} \cup (R_1)_{lg}$ proves the claim stated in Step 1 in the case that $\alpha = \alpha_{sh}^2$. Now suppose that $\alpha = \alpha_{lg}^2$. Setting $\beta := \epsilon_{\ell-1} - \epsilon_\ell$, we get that $\alpha + \beta$ is an extra long root. Now we are done using the same argument as in Type B. Next suppose $\alpha = \alpha_{ex}^2$, then by (2.28), u_t is a weight vector of weight α which is an extra long root. Therefore any extra long root is a weight for \mathcal{U}_t and so the highest weight of \mathcal{U}_t is α_{ex}^1 . This completes the proof of Step 1.

Step 2: We first note that depending on the type of R_2 , α is one of $\epsilon_\ell, 2\epsilon_\ell, \epsilon_\ell + \epsilon_{\ell-1}$ or $\epsilon_{\ell+1} - \epsilon_1$. If $\alpha = \epsilon_\ell + \epsilon_{\ell-1}$, then either R_2 is of type

B or D and by Step 1, the set of nonzero weights of \mathcal{U} coincides with R^\times , or R is of type C or BC and the set of nonzero weights of \mathcal{U} coincides with $(R_{sdiv})_{sh}$. In both cases, using induction on $r \in \mathbb{N} \setminus \{0\}$, one can see that

$$(2.30) \quad \begin{aligned} & \text{if } 1 \leq m_1, \dots, m_r \leq n \text{ and for each } 1 \leq p \leq r, \alpha_{m_p} + \dots + \\ & \alpha_{m_1} + \alpha \text{ is a weight for } \mathcal{U}, \text{ then } \{m_1, \dots, m_r\} \subseteq \{\ell, \dots, n\} \\ & \text{and } \alpha_{m_r} + \dots + \alpha_{m_1} + \alpha = \epsilon_q + \epsilon_{q'} \text{ for some } \ell - 1 \leq q \neq q' \leq n. \end{aligned}$$

Also if $\alpha = \epsilon_{\ell+1} - \epsilon_1, \epsilon_\ell, 2\epsilon_\ell$, one can see that

$$(2.31) \quad \begin{aligned} & \text{if } r \text{ is a positive integer and } 1 \leq m_1, \dots, m_r \leq n \text{ are} \\ & \text{such that for each } 1 \leq p \leq r, \alpha_{m_p} + \dots + \alpha_{m_1} + \alpha \text{ is} \\ & \text{a weight for } \mathcal{U}, \text{ then } \{m_1, \dots, m_r\} \subseteq \{\ell + 1, \dots, n\}. \end{aligned}$$

Now suppose that $0 \neq u \in \mathcal{U}_\alpha$, we shall show that u is a scalar multiple of v . Since $u \in \mathcal{U}$, by (2.27), u is written as a linear combination of weight vectors of the form $f_{i_t} \cdots f_{i_1} e_{j_s} \cdots e_{j_1} \cdot v$, $t, s \in \mathbb{N}$, $1 \leq i_1, \dots, i_t, j_1, \dots, j_s \leq n$. So without loss of generality, we suppose

$$u = f_{i_t} \cdots f_{i_1} \cdot e_{j_s} \cdots e_{j_1} \cdot v$$

where $t, s \in \mathbb{N}$, and $1 \leq i_1, \dots, i_t, j_1, \dots, j_s \leq n$. Since u is of weight α , we get that $\alpha + \alpha_{j_1} + \dots + \alpha_{j_s} - \alpha_{i_1} - \dots - \alpha_{i_t} = \alpha$. This implies that

$$(2.32) \quad s = t \quad \text{and} \quad (j_1, \dots, j_s) = (\sigma(i_1), \dots, \sigma(i_t))$$

for a permutation σ of $\{i_1, \dots, i_t\}$. We note that α is an element of R_2^+ and so it is written as a linear combination of $\{\alpha_i \mid 1 \leq i \leq \ell\}$ with nonnegative rational coefficients not all equal to zero. Now since $\{\alpha_i \mid 1 \leq i \leq n\}$ is a base of $(R_1)_{sdiv}$, $\alpha - \alpha_j$ ($\ell + 1 \leq j \leq n$) is not a root of R_1 and so it is not an element of Λ_1 . Therefore

$$(2.33) \quad f_j \cdot v = 0, \quad \ell + 1 \leq j \leq n.$$

Now this implies that

$$(2.34) \quad \begin{aligned} f_j \cdot e_j \cdot v &= e_j \cdot f_j \cdot v - h_j \cdot v = 0 - h_j \cdot v \in \mathbb{F}v, \\ & (\ell + 1 \leq j \leq n). \end{aligned}$$

We also note that as v is a highest vector of \mathcal{G}_2 -module \mathcal{W} , $e_j \cdot v = 0$ for $1 \leq j \leq \ell$. Therefore one gets that

$$(2.35) \quad j_1 \in \{\ell + 1, \dots, n\} \text{ provided that } s \neq 0.$$

Now we are ready to prove that u is a scalar multiple of v . If $s = 0$, there is nothing to prove. So we suppose $s \geq 1$, and use induction on s to prove. If $s = 1$, we get the result appealing (2.32), (2.35) and (2.34). Now suppose $s > 1$. If $\alpha = \epsilon_\ell + \epsilon_{\ell-1}$, then using (2.30) together with (2.32), we get that $i_1 \in \{\ell, \dots, n\}$. This together with (2.33) and the fact that $2\epsilon_{\ell-1}$ is not a weight for \mathcal{U} implies that $f_{i_1} \cdot v = 0$. Next take $1 \leq k_1 < \dots < k_r \leq s$ to be the only indices with $j_{k_1} = \dots = j_{k_r} = i_1$ and use Lemma 2.25 to get

$$f_{i_t} \cdots f_{i_1} \cdot e_{j_s} \cdots e_{j_1} \cdot v \in \sum_{q=1}^r \mathbb{F} f_{i_t} \cdots f_{i_2} \cdot e_{j_s} \cdots e_{j_{k_r}} \cdots \widehat{e_{j_{k_q}}} \cdots e_{j_{k_1}} \cdots e_{j_1} \cdot v.$$

Now induction hypothesis completes the proof of this step in the case that $\alpha = \epsilon_{\ell-1} + \epsilon_\ell$. Next suppose $\alpha \in \{\epsilon_\ell, 2\epsilon_\ell, \epsilon_{\ell+1} - \epsilon_1\}$, then using (2.31) together with (2.32), (2.33), Lemma 2.25 and the same argument as before, we get the result.

Step 3: It is immediate using Step 2 together with (2.28). \square

3. ROOT GRADED LIE ALGEBRAS

The structure of Lie algebras graded by an irreducible finite root system have been studied in [6], [5], [2], [11], [3] and [7]. A Lie algebra \mathcal{L} graded by an irreducible finite root system R contains a finite dimensional split simple Lie algebra \mathcal{G} and with respect to a splitting Cartan subalgebra, it is equipped with a weight space decomposition whose set of weights is contained in R . This feature allows us to decompose \mathcal{L} into finite dimensional irreducible \mathcal{G} -submodules whose set of nonzero weights is R_{sh} , R_{sdiv}^\times or $(R_{sdiv})_{sh}$. Collecting the components of the same highest weight results in the decomposition

$$(3.1) \quad \mathcal{L} = (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D}$$

in which \mathcal{D} is a trivial submodule of \mathcal{L} and \mathcal{S} (resp. \mathcal{V}) is the finite dimensional irreducible \mathcal{G} -module whose set of nonzero weights is $(R_{sdiv})_{sh}$ (resp. R_{sh}). The Lie algebraic structure on \mathcal{L} induces an algebraic structure on $\mathfrak{b} := \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$ which we refer to as the *coordinate algebra* of \mathcal{L} . The Lie bracket on \mathcal{L} can be rewritten using the ingredients involved in describing the product defined on the algebra \mathfrak{b} . In this section, we have a compromising view on the coordinate algebras of root graded subalgebras of \mathcal{L} . We devote this section to two subsections. In the first subsection, we illustrate the structure of a specific Lie algebra which we shall frequently use in the sequel of the paper. In the second subsection, we consider a Lie algebra \mathcal{L} graded by an irreducible finite root system R and for a full irreducible subsystem S of R which is of the same type as R , we take \mathcal{L}^S to be the Lie subalgebra of \mathcal{L} generated by homogeneous spaces in correspondence to S^\times . We show that the coordinate algebra of the Lie subalgebra \mathcal{L}^S , which is an algebra graded by S , does not depend on S . In fact, we prove that the coordinate algebra of \mathcal{L}^S coincides with the coordinate algebra of \mathcal{L} . Moreover, we describe the Lie bracket on \mathcal{L} in terms of the ingredients involved in describing the Lie bracket on \mathcal{L}^S with respect to its coordinate algebra. Our method is based on a type-by-type approach. Since the proofs for different types are quite similar, we go through the proofs in details if R is of type BC and for other types, we just report the results and leave the proofs to the readers.

3.1. A specific Lie algebra. By a *star algebra* $(\mathfrak{a}, *)$, we mean an algebra \mathfrak{a} together with a self-inverting antiautomorphism $*$ which is referred to as an *involution*. We call a quadruple $(\mathfrak{a}, *, \mathcal{C}, f)$ a *coordinate quadruple* if one of the following holds:

- (Type A) \mathfrak{a} is a unital associative algebra, $*$ = $id_{\mathfrak{a}}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type B) $\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}$ where \mathcal{A} is a unital commutative associative algebra and \mathcal{B} is a unital associative \mathcal{A} -module equipped with a symmetric bilinear form and \mathfrak{a} is the corresponding Clifford Jordan algebra, $*$ is a linear transformation fixing the elements of \mathcal{A} and skew fixing the elements of \mathcal{B} , $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type C) \mathfrak{a} is a unital associative algebra, $*$ is an involution, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type D) \mathfrak{a} is a unital commutative associative algebra $*$ = $id_{\mathfrak{a}}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type BC) \mathfrak{a} is a unital associative algebra, $*$ is an involution, \mathcal{C} is a unital associative \mathfrak{a} -module and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is a skew-hermitian form.

Suppose that $(\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple. Denote by \mathcal{A} and \mathcal{B} , the fixed and the skew fixed points of \mathfrak{a} under $*$, respectively. Set $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f) := \mathfrak{a} \oplus \mathcal{C}$ and define

$$(3.2) \quad \begin{aligned} & \cdot : \mathfrak{b} \times \mathfrak{b} \longrightarrow \mathfrak{b} \\ & (\alpha_1 + c_1, \alpha_2 + c_2) \mapsto (\alpha_1 \cdot \alpha_2) + f(c_1, c_2) + \alpha_1 \cdot c_2 + \alpha_2^* \cdot c_1, \end{aligned}$$

for $\alpha_1, \alpha_2 \in \mathfrak{a}$ and $c_1, c_2 \in \mathcal{C}$. Also for $\beta, \beta' \in \mathfrak{b}$, set

$$(3.3) \quad \beta \circ \beta' := \beta \cdot \beta' + \beta' \cdot \alpha \quad \text{and} \quad [\beta, \beta'] := \beta \cdot \beta' - \beta' \cdot \beta$$

and for $c, c' \in \mathcal{C}$, define

$$(3.4) \quad \begin{aligned} \diamond : \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{A}, & (c, c') &\mapsto \frac{f(c, c') - f(c', c)}{2}; & c, c' &\in \mathcal{C}, \\ \heartsuit : \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{B}, & (c, c') &\mapsto \frac{f(c, c') + f(c', c)}{2}; & c, c' &\in \mathcal{C}. \end{aligned}$$

Now suppose that ℓ is a positive integer and for $\alpha, \alpha' \in \mathfrak{a}$ and $c, c' \in \mathcal{C}$, consider the following endomorphisms

$$\begin{aligned}
 (3.5) \quad & d_{\alpha, \alpha'}^{\ell, \mathfrak{b}} : \mathfrak{b} \longrightarrow \mathfrak{b}, \\
 & \beta \mapsto \begin{cases} \frac{1}{\ell+1} [[\alpha, \alpha'], \beta] & X = A_\ell, \beta \in \mathfrak{b}, \\ \alpha'(\alpha\beta) - \alpha(\alpha'\beta) & X = B_\ell, \beta \in \mathfrak{b}, \\ \frac{1}{4\ell} [[\alpha, \alpha'] + [\alpha^*, \alpha'^*], \beta] & X = C_\ell, BC_\ell, \beta \in \mathfrak{a}, \\ \frac{1}{4\ell} ([\alpha, \alpha'] + [\alpha^*, \alpha'^*]) \cdot \beta & X = C_\ell, BC_\ell, \beta \in \mathcal{C}, \\ 0 & X = D_\ell, \beta \in \mathfrak{b}, \end{cases} \\
 & d_{c, c'}^{\ell, \mathfrak{b}} : \mathfrak{b} \longrightarrow \mathfrak{b}, \\
 & \beta \mapsto \begin{cases} \frac{-1}{2\ell} [c \heartsuit c', \beta] & X = BC_\ell, \beta \in \mathfrak{a}, \\ \frac{-1}{2\ell} (c \heartsuit c') \cdot \beta - \frac{1}{2} (f(\beta, c') \cdot c + f(\beta, c) \cdot c') & X = BC_\ell, \beta \in \mathcal{C}, \\ 0 & \text{otherwise,} \end{cases} \\
 & d_{\alpha, c}^{\ell, \mathfrak{b}} := d_{c, \alpha}^{\ell, \mathfrak{b}} := 0, \\
 & d_{\alpha+c, \alpha'+c'}^{\ell, \mathfrak{b}} := d_{\alpha, \alpha'}^{\ell, \mathfrak{b}} + d_{c, c'}^{\ell, \mathfrak{b}}.
 \end{aligned}$$

One can see that for $\beta, \beta' \in \mathfrak{b}$, $d_{\beta, \beta'}^{\ell, \mathfrak{b}} \in \text{Der}(\mathfrak{b})$. Next take K to be a subspace of $\mathfrak{b} \otimes \mathfrak{b}$ spanned by

$$\begin{aligned}
 & \alpha \otimes c, \quad c \otimes \alpha, \quad a \otimes b, \\
 & \alpha \otimes \alpha' + \alpha' \otimes \alpha, \quad c \otimes c' - c' \otimes c, \\
 & (\alpha \cdot \alpha') \otimes \alpha'' + (\alpha'' \cdot \alpha) \otimes \alpha' + (\alpha' \cdot \alpha'') \otimes \alpha, \\
 & f(c, c') \otimes \alpha + (\alpha^* \cdot c') \otimes c - (\alpha \cdot c) \otimes c'
 \end{aligned}$$

for $\alpha, \alpha', \alpha'' \in \mathfrak{a}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c, c' \in \mathcal{C}$. Then $(\mathfrak{b} \otimes \mathfrak{b})/K$ is a Lie algebra under the Lie bracket

$$(3.6) \quad [(\beta_1 \otimes \beta_2) + K, (\beta'_1 \otimes \beta'_2) + K]_\ell = ((d_{\beta_1, \beta_2}^{\ell, \mathfrak{b}}(\beta'_1) \otimes \beta'_2) + K) + (\beta'_1 \otimes d_{\beta_1, \beta_2}^{\ell, \mathfrak{b}}(\beta'_2)) + K$$

for $\beta_1, \beta_2, \beta'_1, \beta'_2 \in \mathfrak{b}$ (see [3, Proposition 5.23] and [2]). We denote this Lie algebra by $\{\mathfrak{b}, \mathfrak{b}\}_\ell$ (or $\{\mathfrak{b}, \mathfrak{b}\}$ if there is no confusion) and for $\beta_1, \beta_2 \in \mathfrak{b}$, we denote $(\beta_1 \otimes \beta_2) + K$ by $\{\beta_1, \beta_2\}_\ell$ (or $\{\beta_1, \beta_2\}$ if there is no confusion). We recall the *full skew-dihedral homology group*

$$FH(\mathfrak{b}) := \left\{ \sum_{i=1}^n \{\beta_i, \beta'_i\}_\ell \in \{\mathfrak{b}, \mathfrak{b}\}_\ell \mid \sum_{i=1}^n d_{\beta_i, \beta'_i}^{\ell, \mathfrak{b}} = 0 \right\}$$

of \mathfrak{b} (with respect to ℓ) from [3] and [2] and note that it is a subset of the center of $\{\mathfrak{b}, \mathfrak{b}\}_\ell$. For $\beta_1 = a_1 + b_1 + c_1 \in \mathfrak{b}$ and $\beta_2 = a_2 + b_2 + c_2 \in \mathfrak{b}$ with $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$, set

$$(3.7) \quad \beta_{\beta_1, \beta_2}^* := [a_1, a_2] + [b_1, b_2] - c_1 \heartsuit c_2.$$

We say a subset \mathcal{K} of the full skew-dihedral homology group of \mathfrak{b} satisfies the “uniform property on \mathfrak{b} ” if for $\beta_1, \beta'_1, \dots, \beta_n, \beta'_n \in \mathfrak{b}$, $\sum_{i=1}^n \{\beta_i, \beta'_i\}_\ell \in \mathcal{K}$ implies that $\sum_{i=1}^n \beta_{\beta_i, \beta'_i}^* = 0$.

Remark 3.8. Suppose that ℓ, ℓ' are two positive integers. If \mathcal{K} is a subset of the full skew-dihedral homology group of $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ with respect to ℓ satisfying the uniform property on $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$, it is a subset of the full skew-dihedral homology group of $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ with respect to ℓ' satisfying the uniform property on $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$. In other words, the uniform property on $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ dose not depend on ℓ .

3.2. Lie algebras graded by a finite root system. In this work, we study root graded Lie algebras in the following sense:

Definition 3.9. Suppose that R is an irreducible locally finite root system. We say a Lie algebra \mathcal{L} is an R -graded Lie algebra with graded pair $(\mathcal{G}, \mathcal{H})$ if the followings are satisfied:

- i) \mathcal{G} is a locally finite split simple Lie subalgebra of \mathcal{L} with splitting Cartan subalgebra \mathcal{H} and corresponding root system R_{div} .
- ii) \mathcal{L} has a weight space decomposition $\mathcal{L} = \bigoplus_{\alpha \in R} \mathcal{L}_\alpha$ with respect to \mathcal{H} via the adjoint representation.
- iii) $\mathcal{L}_0 = \sum_{\alpha \in R^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.

The following lemma easily follows from Lemma 2.16.

Lemma 3.10. Suppose that R is an irreducible locally finite root system and \mathcal{L} is a Lie algebra graded by R with grading pair $(\mathcal{G}, \mathcal{H})$. Let S be an irreducible full subsystem of R and set

$$(3.11) \quad \begin{aligned} \mathcal{L}^S &:= \sum_{\alpha \in S^\times} \mathcal{L}_\alpha \oplus \sum_{\alpha \in S^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}], \\ \mathcal{G}^S &:= \sum_{\alpha \in S_{\text{div}}^\times} \mathcal{G}_\alpha \oplus \sum_{\alpha \in S_{\text{div}}^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]. \end{aligned}$$

Then \mathcal{L}^S is an S -graded Lie subalgebra of \mathcal{L} with grading pair $(\mathcal{G}^S, \mathcal{H}^S := \mathcal{H} \cap \mathcal{G}^S)$.

Before going through the main body of this subsection, we want to fix a notation. If A is a subspace of a vector space V_1 and B is a subspace of a vector space V_2 , by a conventional notation, we take $A \dot{\otimes} B$ to be the vector subspace of $V_1 \otimes V_2$ spanned by $a \otimes b$ for $a \in A$ and $b \in B$.

3.2.1. Type BC. Suppose that I is a nonempty index set of cardinality $m_n := n > 3$ and I_0 is a nonempty subset of I of cardinality $m_\ell := \ell > 3$. Take $\mathcal{V} := \mathcal{V}^n$ to be a vector space with a basis $\{v_i \mid i \in I \cup \bar{I}\}$ equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) as in (2.13). Set $\mathcal{G}^n := \mathfrak{sp}(I)$ and take $\mathcal{S} := \mathcal{S}^n$ to be as in (2.22). Consider (2.23), (2.14) and set

$$\mathcal{V}^\ell := \mathcal{V}_{I_0}, \quad \mathcal{G}^\ell := \mathcal{G}_{I_0}, \quad \mathcal{S}^\ell := \mathcal{S}_{I_0}.$$

We take $Id_{\mathcal{V}}$ to be the identity map on \mathcal{V} and define the linear endomorphism $Id_{\mathcal{V}^\ell}$ on \mathcal{V} by

$$Id_{\mathcal{V}^\ell} : \mathcal{V} \longrightarrow \mathcal{V} \\ v_i \mapsto v_i, v_{\bar{i}} \mapsto v_{\bar{i}}, v_j \mapsto 0, v_{\bar{j}} \mapsto 0; \quad (i \in I_0, j \in I \setminus I_0).$$

Also for $\lambda = \ell, n$ and $x, y \in \mathcal{G}^\lambda \oplus \mathcal{S}^\lambda$, set

$$(3.12) \quad x \circ_\lambda y := xy + yx - (1/m_\lambda)tr(xy)Id_{\mathcal{V}^\lambda}.$$

Next for $u, v \in \mathcal{V}$, define

$$(3.13) \quad [u, v] : \mathcal{V} \longrightarrow \mathcal{V}; \quad w \mapsto \frac{1}{2}((v, w)u + (w, u)v) + \frac{1}{2\ell}(u, v)Id_{\mathcal{V}^\ell}(w); \quad w \in \mathcal{V}, \\ u \circ v : \mathcal{V} \longrightarrow \mathcal{V}; \quad w \mapsto \frac{1}{2}((v, w)u + (u, w)v); \quad w \in \mathcal{V}, \\ [u, v]_n : \mathcal{V} \longrightarrow \mathcal{V}; \quad w \mapsto \frac{1}{2}((v, w)u + (w, u)v) + \frac{1}{2n}(u, v)Id_{\mathcal{V}}(w); \quad w \in \mathcal{V}.$$

One can easily see that up to isomorphism

$$\mathcal{G}^\ell = \text{span}\{u \circ v \mid u, v \in \mathcal{V}^\ell\}, \quad \mathcal{S}^\ell = \text{span}\{[u, v] \mid u, v \in \mathcal{V}^\ell\}, \\ \mathcal{G}^n = \text{span}\{u \circ v \mid u, v \in \mathcal{V}^n\}, \quad \mathcal{S}^n = \text{span}\{[u, v]_n \mid u, v \in \mathcal{V}^n\}.$$

Suppose that R is an irreducible finite root system of type BC_I and S is the irreducible full subsystem of R of type BC_{I_0} . Suppose that \mathcal{L} is an R -graded Lie algebra with grading pair $(\mathfrak{g}, \mathfrak{h})$ and take $\mathcal{L}^S, \mathfrak{g}^S$ and \mathfrak{h}^S to be as in Lemma 3.10. In order to simplify the using of the notations, we set

$$(3.14) \quad \mathcal{L}^n := \mathcal{L}, \quad \mathcal{L}^\ell := \mathcal{L}^S, \quad [u, v]_\ell := [u, v]; \quad (u, v \in \mathcal{V}).$$

One knows that as a \mathfrak{g}^S -module, \mathcal{L}^ℓ can be decomposed into finite dimensional irreducible \mathfrak{g}^S -submodules, each of which is a finite dimensional irreducible \mathfrak{g}^S -module with highest weight contained in S . Take

$$(3.15) \quad \mathcal{L}^\ell = \mathcal{L}^S = \bigoplus_{i \in \mathcal{I}_0} \mathfrak{g}_i \oplus \bigoplus_{j \in \mathcal{J}_0} \mathfrak{s}_j \oplus \bigoplus_{t \in \mathcal{T}_0} V_t \oplus E$$

to be the decomposition of \mathcal{L}^ℓ into finite dimensional irreducible \mathfrak{g}^S -modules in which $\mathcal{I}_0, \mathcal{J}_0, \mathcal{T}_0$ are (possibly empty) index sets and for $i \in \mathcal{I}_0, j \in \mathcal{J}_0$, and $t \in \mathcal{T}_0$, \mathfrak{g}_i is isomorphic to $\mathfrak{g}^S (\simeq \mathcal{G}^\ell)$, \mathfrak{s}_j is isomorphic to \mathcal{S}^ℓ , V_t is isomorphic to \mathcal{V}^ℓ and E is a trivial \mathfrak{g}^S -submodule.

Lemma 3.16. *Use the notation as in the text and consider $\mathcal{L} = \mathcal{L}^n$ as a \mathfrak{g} -module. Then there exist index sets $\mathcal{I}, \mathcal{J}, \mathcal{T}$ with $\mathcal{I}_0 \subseteq \mathcal{I}, \mathcal{J}_0 \subseteq \mathcal{J}, \mathcal{T}_0 \subseteq \mathcal{T}$, and a class $\{\mathcal{D}_n, \mathcal{G}_i, \mathcal{S}_j, \mathcal{V}_t \mid i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T}\}$ of finite dimensional \mathfrak{g} -submodules of \mathcal{L} such that*

- \mathcal{D}_n is a trivial \mathfrak{g} -module, \mathcal{G}_i is isomorphic to \mathfrak{g} , \mathcal{S}_j is isomorphic to S , and \mathcal{V}_t is isomorphic to \mathcal{V} , for $i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T}$,
- $\mathfrak{g}_i \subseteq \mathcal{G}_i, \mathfrak{s}_j \subseteq \mathcal{S}_j, V_t \subseteq \mathcal{V}_t$ ($i \in \mathcal{I}_0, j \in \mathcal{J}_0, t \in \mathcal{T}_0$),
- $\mathcal{L}^n = \mathcal{L} = \bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathcal{S}_j \oplus \bigoplus_{t \in \mathcal{T}} \mathcal{V}_t \oplus \mathcal{D}_n$.

We make a convention that we refer to

$$(3.17) \quad (\mathcal{I}, \mathcal{J}, \mathcal{T}, \{\mathfrak{g}_i\}, \{\mathcal{G}_i\}, \{\mathfrak{s}_j\}, \{\mathcal{S}_j\}, \{V_t\}, \{\mathcal{V}_t\}, E, \mathcal{D}_n)$$

as an (R, S) -datum for the pair $(\mathcal{L}^n, \mathcal{L}^\ell)$.

Proof. For $i \in \mathcal{I}_0$, by Proposition 2.26, the \mathcal{G} -submodule \mathcal{G}_i of \mathcal{L} generated by \mathfrak{g}_i is a finite dimensional \mathcal{G} -module isomorphic to \mathcal{G} . For $\alpha \in S_{div}^\times$ and $0 \neq x \in (\mathfrak{g}_i)_\alpha \subseteq \mathcal{L}_\alpha$, we have $x \in \mathcal{L}_\alpha \cap \mathfrak{g}_i \subseteq \mathcal{L}_\alpha \cap \mathcal{G}_i = (\mathcal{G}_i)_\alpha$. Now as $\dim(\mathcal{G}_i)_\alpha = \dim(\mathfrak{g}_i)_\alpha = 1$, we get

$$(3.18) \quad (\mathfrak{g}_i)_\alpha = (\mathcal{G}_i)_\alpha; \quad \alpha \in S_{div}^\times.$$

Claim 1. The sum $\sum_{i \in \mathcal{I}_0} \mathcal{G}_i$ is a direct sum: Suppose that i_0, i_1, \dots, i_n are distinct elements of \mathcal{I}_0 and $0 \neq x \in \mathcal{G}_{i_0} \cap \sum_{t=1}^n \mathcal{G}_{i_t}$. Then as \mathcal{G}_{i_0} is an irreducible \mathcal{G} -module, we get that

$$\mathcal{G}_{i_0} \subseteq \sum_{t=1}^n \mathcal{G}_{i_t}.$$

This together with (3.18) implies that for $\alpha \in S_{div}^\times \subseteq R_{div}^\times$,

$$(\mathfrak{g}_{i_0})_\alpha = (\mathcal{G}_{i_0})_\alpha \subseteq \sum_{t=1}^n (\mathcal{G}_{i_t})_\alpha = \sum_{t=1}^n (\mathfrak{g}_{i_t})_\alpha$$

which contradicts the fact that $\sum_{i \in \mathcal{I}_0} \mathfrak{g}_i$ is direct. This completes the proof of Claim 1.

Now for $j \in \mathcal{J}_0$ and $t \in \mathcal{T}_0$, take \mathcal{S}_j and \mathcal{V}_t to be the finite dimensional irreducible \mathcal{G} -submodules of \mathcal{L} generated by \mathfrak{s}_j and V_t respectively. Using the same argument as above, one can see that the summations $\sum_{j \in \mathcal{J}_0} \mathcal{S}_j$ and $\sum_{t \in \mathcal{T}_0} \mathcal{V}_t$ are direct. Set

$$\mathcal{G}(n) := \oplus_{i \in \mathcal{I}_0} \mathcal{G}_i, \quad \mathcal{S}(n) := \oplus_{j \in \mathcal{J}_0} \mathcal{S}_j, \quad \mathcal{V}(n) := \oplus_{t \in \mathcal{T}_0} \mathcal{V}_t.$$

We note that $\mathcal{G}(n)$ (resp. $\mathcal{S}(n)$ and $\mathcal{V}(n)$) is a \mathcal{G} -submodule of \mathcal{L} whose set of weights is R_{div} (resp. $R_{lg} \cup \{0\}$ and R_{sh}).

Claim 2. For $\alpha \in R_{lg}$, $x \in \mathcal{G}(n)_\alpha$, and $y \in \mathcal{S}(n)_\alpha$, we have $x+y=0$ if and only if $x=y=0$: Suppose that $x+y=0$. Since $x \in \mathcal{G}(n)_\alpha = \sum_{i \in \mathcal{I}_0} (\mathcal{G}_i)_\alpha$, we get $x = \sum_{i \in \mathcal{I}_0} x_i$ with finitely many nonzero terms $x_i \in (\mathcal{G}_i)_\alpha$, for $i \in \mathcal{I}_0$. Similarly $y = \sum_{j \in \mathcal{J}_0} y_j$ with finitely many nonzero terms $y_j \in (\mathcal{S}_j)_\alpha$, for $j \in \mathcal{J}_0$. Now we recall that $\ell, n > 3$ and R, S are root systems of type BC_n and BC_ℓ respectively. This allows us to pick $\beta_1, \beta_2 \in R_{lg}$ such that $\beta := \alpha + \beta_1 + \beta_2 \in S_{lg}$ and $\alpha + \beta_1 \in R_{lg}$. Fix $a_1 \in \mathcal{G}_{\beta_1}$ and $a_2 \in \mathcal{G}_{\beta_2}$. Using Lemma 2.24, we get that

$$(3.19) \quad \begin{aligned} & \text{if } x_i \text{ (} i \in \mathcal{I}_0 \text{) is nonzero, then } a_2 \cdot a_1 \cdot x_i \text{ is a nonzero} \\ & \text{element of } (\mathcal{G}_i)_\beta = (\mathfrak{g}_i)_\beta \text{ and similarly if } j \in \mathcal{J}_0 \text{ and} \\ & y_j \neq 0, a_2 \cdot a_1 \cdot y_j \text{ is a nonzero element of } (\mathcal{S}_j)_\beta = (\mathfrak{s}_j)_\beta. \end{aligned}$$

Now since $x+y=0$, we get that $\sum_{i \in \mathcal{I}_0} x_i = -\sum_{j \in \mathcal{J}_0} y_j$ which in turn implies that $\sum_{i \in \mathcal{I}_0} a_2 \cdot a_1 \cdot x_i = -\sum_{j \in \mathcal{J}_0} a_2 \cdot a_1 \cdot y_j$. But the right hand side is

an element of $\oplus_{j \in \mathcal{J}_0} (\mathfrak{s}_j)_\beta$ and the left hand side is an element of $\oplus_{i \in \mathcal{I}_0} (\mathfrak{g}_i)_\beta$. Therefore $a_2 \cdot a_1 \cdot x_i = 0$ and $a_2 \cdot a_1 \cdot y_j = 0$ for $i \in \mathcal{I}_0$ and $j \in \mathcal{J}_0$. This together with (3.19) implies that for $i \in \mathcal{I}_0$ and $j \in \mathcal{J}_0$, $x_i = 0$ and $y_j = 0$. This completes the proof of Claim 2.

Claim 3. For $x \in \mathcal{G}(n)_0$ and $y \in \mathcal{S}(n)_0$, $x+y=0$ if and only if $x=y=0$: Suppose that $x+y=0$ and $x \neq 0$. Since $\mathcal{G}(n)_0 = \sum_{i \in \mathcal{I}_0} (\mathcal{G}_i)_0$, we have $x = \sum_{i \in \mathcal{I}_0} x_i$ with finitely many nonzero terms $x_i \in (\mathcal{G}_i)_0$, $i \in \mathcal{I}_0$. Fix $t \in \mathcal{I}_0$ such that $x_t \neq 0$. Since x_t is a nonzero element of the irreducible nontrivial \mathfrak{g} -module \mathcal{G}_t , there is $\alpha \in R^\times$ and $0 \neq a \in \mathfrak{g}_\alpha$ such that $a \cdot x_t \neq 0$. We note that $x \in \mathcal{G}(n)_0$ and $y \in \mathcal{S}(n)_0$, therefore we have $a \cdot x \in \mathcal{G}(n)_\alpha$ and $a \cdot y \in \mathcal{S}(n)_\alpha$. Now as $0 = a \cdot x + a \cdot y$, Claim 2 together with the fact that the set of weights of $\mathcal{S}(n)$ is $R_{lg} \cup \{0\}$ implies that $a \cdot x = 0$ and $a \cdot y = 0$. So $\sum_{i \in \mathcal{I}_0} a \cdot x_i = 0$. But by Claim 1, $\sum_{i \in \mathcal{I}_0} \mathcal{G}_i$ is a direct sum, so $a \cdot x_t = 0$ which is a contradiction. Therefore $x = 0$ and so $y = 0$ as well. This completes the proof of Claim 3.

Claim 4. The sum $\mathcal{G}(n) + \mathcal{S}(n) + \mathcal{V}(n)$ is a direct sum: Suppose that $x \in \mathcal{G}(n)$, $y \in \mathcal{S}(n)$ and $z \in \mathcal{V}(n)$ are such that $x+y+z=0$. We have $x = \sum_{\alpha \in R_{sdiv}} x_\alpha$ with $x_\alpha \in \mathcal{G}(n)_\alpha \subseteq \mathcal{L}_\alpha$ for $\alpha \in R_{sdiv}$, $y = \sum_{\alpha \in R_{lg} \cup \{0\}} y_\alpha$ with $y_\alpha \in \mathcal{S}(n)_\alpha \subseteq \mathcal{L}_\alpha$ for $\alpha \in R_{lg} \cup \{0\}$, and $z = \sum_{\alpha \in R_{sh}} z_\alpha$ with $z_\alpha \in \mathcal{V}(n)_\alpha \subseteq \mathcal{L}_\alpha$ for $\alpha \in R_{sh}$. Therefore one gets that

$$x_0 + y_0 = 0, \quad z_\alpha = 0, \quad x_\beta + y_\beta = 0, \quad x_\gamma = 0; \\ (\alpha \in R_{sh}, \quad \beta \in R_{lg}, \quad \gamma \in R_{ex}).$$

Now using Claims 2,3, we are done

To complete the proof, we note that as a \mathfrak{g} -module, \mathcal{L} can be decomposed into finite dimensional irreducible \mathfrak{g} -submodules with the set of weights contained in R . Now as $\bigoplus_{i \in \mathcal{I}_0} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}_0} \mathcal{S}_j \oplus \bigoplus_{t \in \mathcal{T}_0} \mathcal{V}_t$ is a submodule of \mathcal{L} , one can find index sets $\mathcal{I}, \mathcal{J}, \mathcal{T}$ with

$$\mathcal{I}_0 \subseteq \mathcal{I}, \quad \mathcal{J}_0 \subseteq \mathcal{J}, \quad \mathcal{T}_0 \subseteq \mathcal{T}$$

and a class $\{\mathcal{D}_n, \mathcal{G}_i, \mathcal{S}_j, \mathcal{V}_t \mid i \in \mathcal{I} \setminus \mathcal{I}_0, j \in \mathcal{J} \setminus \mathcal{J}_0, t \in \mathcal{T} \setminus \mathcal{T}_0\}$ of finite dimensional \mathfrak{g} -submodules such that \mathcal{D}_n is a trivial \mathfrak{g} -module, \mathcal{G}_i is isomorphic to \mathcal{G} ($i \in \mathcal{I} \setminus \mathcal{I}_0$), \mathcal{S}_j is isomorphic to \mathcal{S} ($j \in \mathcal{J} \setminus \mathcal{J}_0$), \mathcal{V}_t is isomorphic to \mathcal{V} ($t \in \mathcal{T} \setminus \mathcal{T}_0$) and

$$\begin{aligned} \mathcal{L} &= \left(\bigoplus_{i \in \mathcal{I}_0} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}_0} \mathcal{S}_j \oplus \bigoplus_{t \in \mathcal{T}_0} \mathcal{V}_t \right) \oplus \left(\bigoplus_{i \in \mathcal{I} \setminus \mathcal{I}_0} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J} \setminus \mathcal{J}_0} \mathcal{S}_j \oplus \bigoplus_{t \in \mathcal{T} \setminus \mathcal{T}_0} \mathcal{V}_t \oplus \mathcal{D}_n \right) \\ &= \bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathcal{S}_j \oplus \bigoplus_{t \in \mathcal{T}} \mathcal{V}_t \oplus \mathcal{D}_n. \end{aligned}$$

This completes the proof. \square

From now on, we use the data appeared in the (R, S) -datum (3.17). We take \mathcal{A}_n to be a vector space with a basis $\{a_i \mid i \in \mathcal{I}\}$, \mathcal{B}_n to be a vector space with a basis $\{b_j \mid j \in \mathcal{J}\}$ and \mathcal{C}_n to be a vector space with a basis $\{c_t \mid t \in \mathcal{T}\}$. Then as a \mathcal{G}^n -module, \mathcal{L} can be identified with

$$(3.20) \quad (\mathcal{G}^n \otimes \mathcal{A}_n) \oplus (\mathcal{S}^n \otimes \mathcal{B}_n) \oplus (\mathcal{V}^n \otimes \mathcal{C}_n) \oplus \mathcal{D}_n.$$

Take

$$(3.21) \quad \varphi : \mathcal{L} \longrightarrow (\mathcal{G}^n \otimes \mathcal{A}_n) \oplus (\mathcal{S}^n \otimes \mathcal{B}_n) \oplus (\mathcal{V}^n \otimes \mathcal{C}_n) \oplus \mathcal{D}_n$$

to be the canonical identification. Next define \mathcal{A}_ℓ to be the vector subspace of \mathcal{A}_n spanned by $\{a_i \mid i \in \mathcal{I}_0\}$, \mathcal{B}_ℓ to be the vector subspace of \mathcal{B}_n spanned by $\{b_j \mid j \in \mathcal{J}_0\}$ and \mathcal{C}_ℓ to be the vector subspace of \mathcal{C}_n spanned by $\{c_t \mid t \in \mathcal{T}_0\}$. Then it follows from (3.15) that as a \mathcal{G}^ℓ -module, $\mathcal{L}^\ell = \mathcal{L}^S$ can be identified with

$$(3.22) \quad (\mathcal{G}^\ell \dot{\otimes} \mathcal{A}_\ell) \oplus (\mathcal{S}^\ell \dot{\otimes} \mathcal{B}_\ell) \oplus (\mathcal{V}^\ell \dot{\otimes} \mathcal{C}_\ell) \oplus \mathcal{D}_\ell$$

where $\mathcal{D}_\ell := \varphi(E)$. In what follows using [3, Thm. 2.48], for $\mu = \ell, n$, we give the algebraical structure of \mathcal{L}^μ in terms of the ingredients involved in the decomposition of \mathcal{L}^μ into finite dimensional irreducible \mathcal{G}^μ -modules. Set $\mathfrak{a}_\mu := \mathcal{A}_\mu \oplus \mathcal{B}_\mu$. Then there are a bilinear map $\cdot_\mu : \mathfrak{a}_\mu \times \mathfrak{a}_\mu \longrightarrow \mathfrak{a}_\mu$ and a linear map $*_\mu : \mathfrak{a}_\mu \longrightarrow \mathfrak{a}_\mu$ such that $(\mathfrak{a}_\mu, \cdot_\mu)$ is a unital associative algebra and $*_\mu$ is an involution on \mathfrak{a}_μ with $*_\mu$ -fixed points \mathcal{A}_μ and $*_\mu$ -skew fixed points \mathcal{B}_μ . Also there is a bilinear map $\cdot_\mu : \mathfrak{a}_\mu \times \mathcal{C}_\mu \longrightarrow \mathcal{C}_\mu$ such that $(\mathcal{C}_\mu, \cdot_\mu)$ is a left unital associative \mathfrak{a}_μ -module equipped with a skew-hermitian form $f_\mu : \mathcal{C}_\mu \times \mathcal{C}_\mu \longrightarrow \mathfrak{a}_\mu$. Take $\mathfrak{b}_\mu := \mathfrak{b}(\mathfrak{a}_\mu, *_\mu, \mathcal{C}_\mu, f_\mu)$ to be defined as in Subsection 3.1 and set $\cdot_\mu, \circ_\mu, [\cdot, \cdot]_\mu, \diamond_\mu$ and \heartsuit_μ to be the corresponding features as $\cdot, \circ, [\cdot, \cdot], \diamond$ and \heartsuit defined in Subsection 3.1. Also for $\beta, \beta' \in \mathfrak{b}_\mu$, set $d_{\beta, \beta'}^\mu := d_{\beta, \beta'}^{\mu, \mathfrak{b}_\mu}$. By [3, Theorems 2.48, 5.34], \mathcal{D}_μ is a subalgebra of \mathcal{L}^μ and there is a subspace \mathcal{K}_μ of the full skew-dihedral homology group

$$FH(\mathfrak{b}_\mu) = \left\{ \sum_i \{\beta_i, \beta'_i\}_\mu \mid \sum_i d_{\beta_i, \beta'_i}^\mu = 0 \right\}$$

of \mathfrak{b}_μ such that \mathcal{D}_μ is isomorphic to the quotient algebra $\{\mathfrak{b}_\mu, \mathfrak{b}_\mu\}_\mu / \mathcal{K}_\mu$. For β_1, β_2 , take $\langle \beta_1, \beta_2 \rangle_\mu$ to be the element of \mathcal{D}_μ corresponding to $\{\beta_1, \beta_2\}_\mu + \mathcal{K}_\mu$, then one has $\langle \mathcal{A}_\mu, \mathcal{B}_\mu \rangle_\mu = \langle \mathcal{A}_\mu, \mathcal{C}_\mu \rangle_\mu = \langle \mathcal{B}_\mu, \mathcal{C}_\mu \rangle_\mu = \{0\}$ and $\mathcal{D}_\mu = \langle \mathcal{A}_\mu, \mathcal{A}_\mu \rangle_\mu + \langle \mathcal{B}_\mu, \mathcal{B}_\mu \rangle_\mu + \langle \mathcal{C}_\mu, \mathcal{C}_\mu \rangle_\mu$. Moreover the Lie bracket on \mathcal{L}^μ which is

an extension of the Lie bracket on \mathcal{D}_μ is given by

(3.23)

$$\begin{aligned}
[x \otimes a, y \otimes a'] &= [x, y] \otimes \frac{1}{2}(a \circ_\mu a') + (x \circ_\mu y) \otimes \frac{1}{2}[a, a']_\mu + tr(xy)\langle a, a' \rangle_\mu, \\
[x \otimes a, s \otimes b] &= (x \circ_\mu s) \otimes \frac{1}{2}[a, b]_\mu + [x, s] \otimes \frac{1}{2}(a \circ_\mu b) = -[s \otimes b, x \otimes a], \\
[s \otimes b, t \otimes b'] &= [s, t] \otimes \frac{1}{2}(b \circ_\mu b') + (s \circ_\mu t) \otimes \frac{1}{2}[b, b']_\mu + tr(st)\langle b, b' \rangle_\mu, \\
[x \otimes a, u \otimes c] &= xu \otimes a \cdot_\mu c = -[u \otimes c, x \otimes a], \\
[s \otimes b, u \otimes c] &= su \otimes b \cdot_\mu c = -[u \otimes c, s \otimes b], \\
[u \otimes c, v \otimes c'] &= (u \circ v) \otimes (c \circ_\mu c') + [u, v]_\mu \otimes (c \heartsuit_\mu c') + (u, v)\langle c, c' \rangle_\mu, \\
[\langle \beta, \beta' \rangle_\mu, x \otimes a] &= x \otimes d_{\beta, \beta'}^\mu(a) = -[x \otimes a, \langle \beta, \beta' \rangle_\mu], \\
[\langle \beta, \beta' \rangle_\mu, s \otimes b] &= s \otimes d_{\beta, \beta'}^\mu(b) = -[s \otimes b, \langle \beta, \beta' \rangle_\mu], \\
[\langle \beta, \beta' \rangle_\mu, u \otimes c] &= u \otimes d_{\beta, \beta'}^\mu(c) = -[u \otimes c, \langle \beta, \beta' \rangle_\mu], \\
[\langle \beta_1, \beta_2 \rangle_\mu, \langle \beta'_1, \beta'_2 \rangle_\mu] &= \langle d_{\beta_1, \beta_2}^\mu(\beta'_1), \beta'_2 \rangle_\mu + \langle \beta'_1, d_{\beta_1, \beta_2}^\mu(\beta'_2) \rangle_\mu,
\end{aligned}$$

for $x, y \in \mathcal{G}^\mu$, $s, t \in \mathcal{S}^\mu$, $u, v \in \mathcal{V}^\mu$, $a, a' \in \mathcal{A}_\mu$, $b, b' \in \mathcal{B}_\mu$, $c, c' \in \mathcal{C}_\mu$ and $\beta, \beta' \in \mathfrak{b}_\mu$.

Lemma 3.24. *We have $\mathcal{I} = \mathcal{I}_0$, $\mathcal{J} = \mathcal{J}_0$ and $\mathcal{T} = \mathcal{T}_0$.*

Proof. It follows from (3.23), (3.20) and (3.22) that

$$\mathcal{L}_\alpha = \begin{cases} \mathcal{V}_\alpha^n \dot{\otimes} \mathcal{C}_n & \text{if } \alpha \in R_{sh} \\ (\mathcal{G}_\alpha^n \dot{\otimes} \mathcal{A}_n) \oplus (\mathcal{S}_\alpha^n \dot{\otimes} \mathcal{B}_n) & \text{if } \alpha \in R_{lg} \\ \mathcal{G}_\alpha^n \dot{\otimes} \mathcal{A}_n & \text{if } \alpha \in R_{ex} \end{cases}$$

and

$$(\mathcal{L}^S)_\alpha = \begin{cases} (\mathcal{V}^\ell)_\alpha \dot{\otimes} \mathcal{C}_\ell & \text{if } \alpha \in S_{sh} \\ (\mathcal{G}_\alpha^\ell \dot{\otimes} \mathcal{A}_\ell) \oplus (\mathcal{S}_\alpha^\ell \dot{\otimes} \mathcal{B}_\ell) & \text{if } \alpha \in S_{lg} \\ \mathcal{G}_\alpha^\ell \dot{\otimes} \mathcal{A}_\ell & \text{if } \alpha \in S_{ex}. \end{cases}$$

Now fix $\alpha \in S_{ex}$, then

$$(\mathcal{G}^\ell)_\alpha \dot{\otimes} \mathcal{A}_\ell = (\mathcal{L}^S)_\alpha = \mathcal{L}_\alpha = \mathcal{G}_\alpha^n \dot{\otimes} \mathcal{A}_n.$$

This together with the fact that $\mathcal{G}_\alpha^\ell = \mathcal{G}_\alpha^n$ is a one dimensional vector space, implies that the vector space \mathcal{A}_ℓ equals the vector space \mathcal{A}_n . In particular we get $\mathcal{I} = \mathcal{I}_0$. Next fix $\alpha \in S_{sh}$, then we have

$$\mathcal{V}_\alpha^\ell \dot{\otimes} \mathcal{C}_\ell = \mathcal{L}_\alpha^S = \mathcal{L}_\alpha = \mathcal{V}_\alpha^n \dot{\otimes} \mathcal{C}_n.$$

This as above, implies that $\mathcal{T} = \mathcal{T}_0$. Finally fix $\alpha \in S_{lg}$, then

$$(\mathcal{G}_\alpha^\ell \dot{\otimes} \mathcal{A}_\ell) \oplus (\mathcal{S}_\alpha^\ell \dot{\otimes} \mathcal{B}_\ell) = \mathcal{L}_\alpha^S = \mathcal{L}_\alpha = (\mathcal{G}_\alpha^n \dot{\otimes} \mathcal{A}_n) \oplus (\mathcal{S}_\alpha^n \dot{\otimes} \mathcal{B}_n).$$

Now as $\mathcal{S}_\alpha^\ell = \mathcal{S}_\alpha^n$ is a one dimensional vector space, $\mathcal{G}_\alpha^\ell = \mathcal{G}_\alpha^n$, $\mathcal{B}_\ell \subseteq \mathcal{B}_n$ and $\mathcal{A}_\ell = \mathcal{A}_n$, we get that the two vector spaces \mathcal{B}_ℓ and \mathcal{B}_n are equal and so $\mathcal{J} = \mathcal{J}_0$. \square

As we have already seen, on the vector space level, we have

$$\mathcal{A}_\ell = \mathcal{A}_n, \mathcal{B}_\ell = \mathcal{B}_n, \mathcal{C}_\ell = \mathcal{C}_n$$

which in turn implies that the vector space \mathfrak{b}_ℓ equals the vector space \mathfrak{b}_n . In the following lemma, we show, in addition, that the algebras \mathfrak{b}_ℓ and \mathfrak{b}_n have the same algebraic structure.

Lemma 3.25. *The algebraic structure of \mathfrak{b}_n coincides with the algebraic structure of \mathfrak{b}_ℓ .*

Proof. Using Lemma 3.24, we set

$$(3.26) \quad \mathcal{A} := \mathcal{A}_\ell = \mathcal{A}_n, \quad \mathcal{B} := \mathcal{B}_\ell = \mathcal{B}_n, \quad \mathcal{C} := \mathcal{C}_\ell = \mathcal{C}_n.$$

Suppose that i, j, k are distinct elements of I_0 . Take $x := e_{i,j} - e_{\bar{j},\bar{i}} \in \mathcal{G}^\ell$ and $y := e_{j,k} - e_{\bar{k},\bar{j}} \in \mathcal{G}^\ell$, then

$$\text{tr}(xy) = 0, \quad [x, y] = e_{i,k} - e_{\bar{k},\bar{i}}, \quad x \circ_n y = x \circ_\ell y = e_{i,k} + e_{\bar{k},\bar{i}}.$$

Now for $a, a' \in \mathcal{A}$, by (3.23), we have

$$\begin{aligned} [x, y] \otimes \frac{1}{2}(a \circ_n a') + (x \circ_n y) \otimes \frac{1}{2}[a, a']_n &= [x \otimes a, y \otimes a'] \\ &= [x, y] \otimes \frac{1}{2}(a \circ_\ell a') + (x \circ_\ell y) \otimes \frac{1}{2}[a, a']_\ell. \end{aligned}$$

This in turn implies that

$$[x, y] \otimes \left(\frac{1}{2}(a \circ_n a') - \frac{1}{2}(a \circ_\ell a') \right) = (x \circ_n y) \otimes \left(\frac{1}{2}[a, a']_\ell - \frac{1}{2}[a, a']_n \right),$$

but the left hand side is an element of $\mathcal{G} \otimes \mathcal{A}$ and the right hand side is an element of $\mathcal{S} \otimes \mathcal{B}$. Therefore as $[x, y] \neq 0$ and $x \circ_n y \neq 0$, we get that

$$\frac{1}{2}[a, a']_\ell - \frac{1}{2}[a, a']_n = 0 \quad \text{and} \quad \frac{1}{2}(a \circ_n a') - \frac{1}{2}(a \circ_\ell a') = 0.$$

This now implies that

$$(3.27) \quad a \cdot_\ell a' = a \cdot_n a'; \quad a, a' \in \mathcal{A}.$$

Next take i and j to be two distinct elements of I_0 . Set $s := e_{i,j} + e_{\bar{j},\bar{i}} \in \mathcal{S}^\ell$ and $x := e_{j,\bar{j}} \in \mathcal{G}^\ell$, then we have

$$\text{tr}(xs) = 0, \quad [x, s] = e_{j,\bar{i}} - e_{i,\bar{j}}, \quad x \circ_\ell s = x \circ_n s = e_{j,\bar{i}} + e_{i,\bar{j}}.$$

Now for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, by (3.23), we have

$$\begin{aligned} (x \circ_\ell s) \otimes \frac{1}{2}[a, b]_\ell + [x, s] \otimes \frac{1}{2}(a \circ_\ell b) &= [x \otimes a, s \otimes b] \\ &= (x \circ_n s) \otimes \frac{1}{2}[a, b]_n + [x, s] \otimes \frac{1}{2}(a \circ_n b). \end{aligned}$$

This implies that

$$(x \circ_\ell s) \otimes \left(\frac{1}{2}[a, b]_\ell - \frac{1}{2}[a, b]_n \right) = [x, s] \otimes \left(\frac{1}{2}(a \circ_n b) - \frac{1}{2}(a \circ_\ell b) \right).$$

Now as before, one gets that

$$\frac{1}{2}[a, b]_\ell - \frac{1}{2}[a, b]_n = 0 \quad \text{and} \quad \frac{1}{2}(a \circ_n b) - \frac{1}{2}(a \circ_\ell b) = 0.$$

This in particular implies that

$$(3.28) \quad a \cdot_\ell b = a \cdot_n b \quad \text{and} \quad b \cdot_\ell a = b \cdot_n a; \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

Finally, for distinct fixed elements i, j, k of I_0 , set $s := e_{i,\bar{j}} - e_{j,\bar{i}}, t = e_{\bar{i},k} - e_{\bar{k},i} \in \mathcal{S}^\ell$. Then

$$tr(st) = 0, [s, t] = -e_{j,k} + e_{\bar{k},\bar{j}}, s \circ_\ell t = s \circ_n t = -e_{j,k} - e_{\bar{k},\bar{j}}.$$

Therefore for $b, b' \in \mathcal{B}$, by (3.23), we have

$$\begin{aligned} ([s, t] \otimes \frac{1}{2}(b \circ_\ell b')) + ((s \circ_\ell t) \otimes \frac{1}{2}[b, b']_\ell) &= [s \otimes b, t \otimes b'] \\ &= ([s, t] \otimes \frac{1}{2}(b \circ_n b')) + ((s \circ_n t) \otimes \frac{1}{2}[b, b']_n). \end{aligned}$$

This implies that

$$[s, t] \otimes (\frac{1}{2}(b \circ_\ell b') - \frac{1}{2}(b \circ_n b')) = (s \circ_n t) \otimes (\frac{1}{2}[b, b']_n - \frac{1}{2}[b, b']_\ell).$$

Therefore we get

$$\frac{1}{2}(b \circ_\ell b') - \frac{1}{2}(b \circ_n b') = 0 \quad \text{and} \quad \frac{1}{2}[b, b']_n - \frac{1}{2}[b, b']_\ell = 0$$

and so we have $b \cdot_\ell b' = b \cdot_n b'$ which together with (3.27) and (3.28) implies that

$$(3.29) \quad \mathfrak{a}_\ell = \mathfrak{a}_n \quad (\text{as two algebras}).$$

Now take $x \in \mathcal{G}^\ell, s \in \mathcal{S}^\ell$ and $u, v \in \mathcal{V}^\ell$ to be such that $xu \neq 0$ and $sv \neq 0$. Then for $a \in \mathcal{A}, b \in \mathcal{B}$ and $c \in \mathcal{C}$, we get using (3.23) that

$$\begin{aligned} xu \otimes a \cdot_\ell c &= [x \otimes a, u \otimes c] = xu \otimes a \cdot_n c, \\ sv \otimes b \cdot_\ell c &= [s \otimes b, v \otimes c] = sv \otimes b \cdot_n c. \end{aligned}$$

This implies that

$$a \cdot_\ell c = a \cdot_n c \quad \text{and} \quad b \cdot_\ell c = b \cdot_n c$$

for $a \in \mathcal{A}, b \in \mathcal{B}$ and $c \in \mathcal{C}$. Therefore we have

$$(3.30) \quad \mathcal{C}_\ell = \mathcal{C}_n \quad (\text{as two } \mathfrak{a}_n\text{-modules}).$$

Now we are done thanks to (3.29) and (3.30). \square

Now using Lemma 3.25, we set

$$\mathfrak{a} := \mathfrak{a}_n = \mathfrak{a}_\ell \quad \text{and} \quad \mathfrak{b} := \mathfrak{b}_\ell = \mathfrak{b}_n.$$

Also for $\beta, \beta' \in \mathfrak{b}$, we take

$$(3.31) \quad \begin{aligned} \beta \cdot \beta' &:= \beta \cdot_n \beta' = \beta \cdot_\ell \beta', \\ [\beta, \beta'] &:= \beta \cdot \beta' - \beta' \cdot \beta, \\ \beta \circ \beta' &:= \beta \cdot \beta' + \beta' \cdot \beta. \end{aligned}$$

Lemma 3.32. For $a, a' \in \mathcal{A}$, and $b, b' \in \mathcal{B}$, we have

$$\begin{aligned}\langle a, a' \rangle_n &= (\frac{-1}{\ell} Id_{\mathcal{V}^\ell} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2} [a, a'] + \langle a, a' \rangle_\ell, \\ \langle b, b' \rangle_n &= (\frac{-1}{\ell} Id_{\mathcal{V}^\ell} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2} [b, b'] + \langle b, b' \rangle_\ell.\end{aligned}$$

Also for $c, c' \in \mathcal{C}$, $f_\ell(c, c') = f_n(c, c')$, $c \diamond_\ell c' = c \diamond_n c'$ and $c \heartsuit_\ell c' = c \heartsuit_n c'$. Moreover we have

$$\begin{aligned}\langle c, c' \rangle_n &= ((\frac{1}{\ell} Id_{\mathcal{V}^\ell} - \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2} c \heartsuit_n c') + \langle c, c' \rangle_\ell \\ &= ((\frac{1}{\ell} Id_{\mathcal{V}^\ell} - \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2} c \heartsuit_\ell c') + \langle c, c' \rangle_\ell.\end{aligned}$$

Proof. Fix $x, y \in \mathcal{G}^\ell$ such that $tr(xy) \neq 0$. For $a, a' \in \mathcal{A}$, consider (3.31) and use (3.23) to get

$$\begin{aligned}([x, y] \otimes \frac{1}{2} (a \circ a')) + ((x \circ_n y) \otimes \frac{1}{2} [a, a']) + tr(xy) \langle a, a' \rangle_n &= [x \otimes a, y \otimes a'] = \\ ([x, y] \otimes \frac{1}{2} (a \circ a')) + ((x \circ_\ell y) \otimes \frac{1}{2} [a, a']) + tr(xy) \langle a, a' \rangle_\ell.\end{aligned}$$

This implies that

$$(\frac{-tr(xy)}{n} Id_{\mathcal{V}} \otimes \frac{1}{2} [a, a']) + tr(xy) \langle a, a' \rangle_n = (\frac{-tr(xy)}{\ell} Id_{\mathcal{V}^\ell} \otimes \frac{1}{2} [a, a']) + tr(xy) \langle a, a' \rangle_\ell.$$

Therefore we have

$$(3.33) \quad \langle a, a' \rangle_n = (\frac{-1}{\ell} Id_{\mathcal{V}^\ell} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2} [a, a'] + \langle a, a' \rangle_\ell; \quad (a, a' \in \mathcal{A}).$$

Next fix $s, t \in \mathcal{S}^\ell$ such that $tr(st) \neq 0$, then for $b, b' \in \mathcal{B}$, by (3.23), we have

$$\begin{aligned}([s, t] \otimes \frac{1}{2} b \circ b') + ((s \circ_n t) \otimes \frac{1}{2} [b, b']) + tr(st) \langle b, b' \rangle_n &= [s \otimes b, t \otimes b'] = \\ ([s, t] \otimes \frac{1}{2} b \circ b') + ((s \circ_\ell t) \otimes \frac{1}{2} [b, b']) + tr(st) \langle b, b' \rangle_\ell.\end{aligned}$$

This implies that

$$(\frac{-tr(st)}{n} Id_{\mathcal{V}} \otimes \frac{1}{2} [b, b']) + tr(st) \langle b, b' \rangle_n = (\frac{-tr(st)}{\ell} Id_{\mathcal{V}^\ell} \otimes \frac{1}{2} [b, b']) + tr(st) \langle b, b' \rangle_\ell$$

which in turn implies that

$$(3.34) \quad \langle b, b' \rangle_n = ((\frac{-1}{\ell} Id_{\mathcal{V}^\ell} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2} [b, b']) + \langle b, b' \rangle_\ell; \quad (b, b' \in \mathcal{B}).$$

Now suppose that i and j are two distinct elements of I_0 . Take $u := v_i$ and $v := v_j$, then $(u, v) = 0$ and so $[u, v]_n = [u, v]$. Therefore for all $c, c' \in \mathcal{C}$, by (3.23), we have

$$\begin{aligned}(u \circ v) \otimes (c \diamond_n c') + [u, v] \otimes (c \heartsuit_n c') &= [u \otimes c, v \otimes c'] \\ &= (u \circ v) \otimes (c \diamond_\ell c') + [u, v] \otimes c \heartsuit_\ell c' .\end{aligned}$$

But $u \circ v \in \mathcal{G}$, $[u, v]$ is a nonzero element of \mathcal{S} , $c \diamond_\ell c', c \diamond_n c' \in \mathcal{A}$ and $c \heartsuit_\ell c', c \heartsuit_n c' \in \mathcal{B}$, so we get that

$$(3.35) \quad c \diamond c' := c \diamond_\ell c' = c \diamond_n c' \quad \text{and} \quad c \heartsuit c' := c \heartsuit_\ell c' = c \heartsuit_n c'; \quad (c, c' \in \mathcal{C}).$$

This in turn implies that

$$(3.36) \quad f(c, c') := f_\ell(c, c') = f_n(c, c'); \quad (c, c' \in \mathcal{C}).$$

Next for an element i of I , take $u := v_i$ and $v = v_{\bar{i}}$. Then for $c, c' \in \mathcal{C}$, by (3.23), we have

$$(u \circ v \otimes c \diamond c') + ([u, v]_n \otimes c \heartsuit c') + \langle c, c' \rangle_n = [u \otimes c, v \otimes c'] = (u \circ v \otimes c \diamond c') + ([u, v]_\ell \otimes c \heartsuit c') + \langle c, c' \rangle_\ell,$$

using which, one concludes that

$$(3.37) \quad \langle c, c' \rangle_n = \langle c, c' \rangle_\ell + \left(\frac{1}{2\ell} Id_{\mathcal{V}} - \frac{1}{2n} Id_{\mathcal{V}^\ell} \right) \otimes c \heartsuit c'; \quad (c, c' \in \mathcal{C}).$$

This completes the proof. \square

Corollary 3.38. *Let $\ell < n$ and suppose that $t \in \mathbb{N}$, $a_i, a'_i \in \mathcal{A}$, $b_i, b'_i \in \mathcal{B}$ and $c_i, c'_i \in \mathcal{C}$ for $1 \leq i \leq t$. Then $\sum_{i=1}^t (\langle a_i, a'_i \rangle_\ell + \langle b_i, b'_i \rangle_\ell + \langle c_i, c'_i \rangle_\ell) = 0$ if and only if $\sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - c_i \heartsuit c'_i) = 0$ and $\sum_{i=1}^t (\langle a_i, a'_i \rangle_n + \langle b_i, b'_i \rangle_n + \langle c_i, c'_i \rangle_n) = 0$.*

Proof. By Lemma 3.32, we have

$$\begin{aligned} & \sum_{i=1}^t (\langle a_i, a'_i \rangle_\ell + \langle b_i, b'_i \rangle_\ell + \langle c_i, c'_i \rangle_\ell) = \\ & \sum_{i=1}^t (\langle a_i, a'_i \rangle_n + \langle b_i, b'_i \rangle_n + \langle c_i, c'_i \rangle_n) - \\ & \left(\frac{-1}{\ell} Id_{\mathcal{V}^\ell} + \frac{1}{n} Id_{\mathcal{V}} \right) \otimes \frac{1}{2} \sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - c_i \heartsuit c'_i). \end{aligned}$$

Now as $\sum_{i=1}^t (\langle a_i, a'_i \rangle_n + \langle b_i, b'_i \rangle_n + \langle c_i, c'_i \rangle_n) \in \mathcal{D}_n$ and $(\frac{-1}{\ell} Id_{\mathcal{V}^\ell} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2} \sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - c_i \heartsuit c'_i) \in \mathcal{S} \otimes \mathcal{B}$, we are done. \square

Remark 3.39. Consider the decomposition (3.15) for $\mathcal{L}^S = \mathcal{L}^\ell$ into finite dimensional irreducible \mathfrak{g}^S -submodules and the decomposition of $\mathcal{L} = \mathcal{L}^n$ into finite dimensional irreducible \mathfrak{g} -submodules as in Lemma 3.16, then contemplating the identification (3.21), we have using Lemma 3.32 that

$$\mathcal{L} = \left(\bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathcal{S}_j \oplus \bigoplus_{t \in \mathcal{T}} \mathcal{V}_t \right) + E.$$

Moreover setting $\langle \beta, \beta' \rangle^n := \varphi^{-1}(\langle \beta, \beta' \rangle_n)$ and $\langle \beta, \beta' \rangle^\ell := \varphi^{-1}(\langle \beta, \beta' \rangle_\ell)$ for $\beta, \beta' \in \mathfrak{b}$, we get that $\{\langle \beta, \beta' \rangle^n \mid \beta, \beta' \in \mathfrak{b}\}$ spans \mathcal{D}_n and that $\{\langle \beta, \beta' \rangle^\ell \mid \beta, \beta' \in \mathfrak{b}\}$ spans E . Furthermore, thanks to Corollary 3.38, for $t \in \mathbb{N}$, $a_i, a'_i \in$

$\mathcal{A}, b_i, b'_i \in \mathcal{B}$ and $c_i, c'_i \in \mathcal{C}$ ($1 \leq i \leq t$), $\sum_{i=1}^t (\langle a_i, a'_i \rangle^\ell + \langle b_i, b'_i \rangle^\ell + \langle c_i, c'_i \rangle^\ell) = 0$ if and only if $\sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - c_i \heartsuit c'_i) = 0$ and $\sum_{i=1}^t (\langle a_i, a'_i \rangle^n + \langle b_i, b'_i \rangle^n + \langle c_i, c'_i \rangle^n) = 0$.

Proposition 3.40. *For $e, f \in \mathcal{G} \cup \mathcal{S}$, set*

$$e \circ f := ef + fe - \frac{\text{tr}(ef)}{\ell} Id_{\mathcal{V}^\ell}.$$

Also for $\beta_1 = a_1 + b_1 + c_1 \in \mathfrak{b}$ and $\beta_2 = a_2 + b_2 + c_2 \in \mathfrak{b}$ with $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$, we recall from (3.7) that $\beta_{\beta_1, \beta_2}^* := [a_1, a_2] + [b_1, b_2] - c_1 \heartsuit c_2$ and set

$$\langle \beta_1, \beta_2 \rangle := \langle \beta_1, \beta_2 \rangle_\ell, \quad \beta_1^* = c_1, \quad \beta_2^* = c_2$$

and take

$$\mathcal{D} := \text{span}\{\langle a, a' \rangle, \langle b, b' \rangle, \langle c, c' \rangle \mid a, a' \in \mathcal{A}, b, b' \in \mathcal{B}, c, c' \in \mathcal{C}\},$$

then contemplating (3.35), we have

$$\begin{aligned} \mathcal{L} &= (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle_n \\ &= ((\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C})) + \mathcal{D} \end{aligned}$$

with the Lie bracket given by

$$\begin{aligned} (3.41) \quad & [x \otimes a, y \otimes a'] = [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + \text{tr}(xy)\langle a, a' \rangle, \\ & [x \otimes a, s \otimes b] = (x \circ s) \otimes \frac{1}{2}[a, b] + [x, s] \otimes \frac{1}{2}(a \circ b) = -[s \otimes b, x \otimes a], \\ & [s \otimes b, t \otimes b'] = [s, t] \otimes \frac{1}{2}(b \circ b') + (s \circ t) \otimes \frac{1}{2}[b, b'] + \text{tr}(st)\langle b, b' \rangle, \\ & [x \otimes a, u \otimes c] = xu \otimes a \cdot c = -[u \otimes c, x \otimes a], \\ & [s \otimes b, u \otimes c] = su \otimes b \cdot c = -[u \otimes c, s \otimes b], \\ & [u \otimes c, v \otimes c'] = (u \circ v) \otimes (c \diamond c') + [u, v] \otimes (c \heartsuit c') + (u, v)\langle c, c' \rangle, \\ & [\langle \beta_1, \beta_2 \rangle, x \otimes a] = \frac{-1}{4\ell}(x \circ Id_{\mathcal{V}^\ell} \otimes [a, \beta_{\beta_1, \beta_2}^*] + [x, Id_{\mathcal{V}^\ell}] \otimes a \circ \beta_{\beta_1, \beta_2}^*), \\ & [\langle \beta_1, \beta_2 \rangle, s \otimes b] = \frac{-1}{4\ell}([s, Id_{\mathcal{V}^\ell}] \otimes [b \circ \beta_{\beta_1, \beta_2}^*] + (s \circ Id_{\mathcal{V}^\ell}) \otimes [b, \beta_{\beta_1, \beta_2}^*] + 2\text{tr}(s Id_{\mathcal{V}^\ell})\langle b, \beta_{\beta_1, \beta_2}^* \rangle), \\ & [\langle \beta_1, \beta_2 \rangle, v \otimes c] = \frac{1}{2\ell} Id_{\mathcal{V}^\ell} v \otimes (\beta_{\beta_1, \beta_2}^* \cdot c) - \frac{1}{2}v \otimes (f(c, \beta_2^*) \cdot \beta_1^* + f(c, \beta_1^*) \cdot \beta_2^*) \\ & [\langle \beta_1, \beta_2 \rangle, \langle \beta'_1, \beta'_2 \rangle] = \langle d_{\beta_1, \beta_2}^\ell(\beta'_1), \beta'_2 \rangle + \langle \beta'_1, d_{\beta_1, \beta_2}^\ell(\beta'_2) \rangle \end{aligned}$$

for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $u, v \in \mathcal{V}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $c, c' \in \mathcal{C}$, $\beta_1, \beta_2, \beta'_1, \beta'_2 \in \mathfrak{b}$.

Proof. Suppose that $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $u, v \in \mathcal{V}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$ and $c, c' \in \mathcal{C}$, then (3.23) (for $\mu = n$) together with Lemma 3.32 implies that

$$\begin{aligned}
 [x \otimes a, y \otimes a'] &= ([x, y] \otimes \frac{1}{2}(a \circ a')) + ((x \circ_n y) \otimes \frac{1}{2}[a, a']) + tr(xy)\langle a, a' \rangle_n \\
 &= ([x, y] \otimes \frac{1}{2}(a \circ a')) + ((x \circ_n y) \otimes \frac{1}{2}[a, a']) \\
 (3.42) \quad &+ tr(xy)(\frac{-1}{\ell}Id_{\mathcal{V}^\ell} + \frac{1}{n}Id_{\mathcal{V}}) \otimes \frac{1}{2}[a, a'] + tr(xy)\langle a, a' \rangle_\ell \\
 &= ([x, y] \otimes \frac{1}{2}(a \circ a')) \\
 &+ (((x \circ_n y) + tr(xy)(\frac{-1}{\ell}Id_{\mathcal{V}^\ell} + \frac{1}{n}Id_{\mathcal{V}})) \otimes \frac{1}{2}[a, a']) + tr(xy)\langle a, a' \rangle_\ell \\
 &= ([x, y] \otimes \frac{1}{2}(a \circ a')) + ((x \circ y) \otimes \frac{1}{2}[a, a']) + tr(xy)\langle a, a' \rangle.
 \end{aligned}$$

Similarly we have

$$[s \otimes b, t \otimes b'] = ([s, t] \otimes \frac{1}{2}(b \circ b')) + ((s \circ t) \otimes \frac{1}{2}[b, b']) + tr(st)\langle b, b' \rangle$$

and

$$[u \otimes c, v \otimes c'] = ((u \circ v) \otimes c \circ c') + ([u, v] \otimes (c \heartsuit c')) + (u, v)\langle c, c' \rangle.$$

Now for $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$, set $b^* := [a_1, a_2] + [b_1, b_2] - c_1 \heartsuit c_2$ and take $s_n := (1/\ell)Id_{\mathcal{V}^\ell} - (1/n)Id_{\mathcal{V}}$. Then for $x \in \mathcal{G}$, $s \in \mathcal{S}$, $v \in \mathcal{V}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$, one can see that

$$[x \otimes a, s_n \otimes b^*] = \frac{1}{2\ell}(x \circ Id_{\mathcal{V}^\ell} \otimes [a, b^*] + [x, Id_{\mathcal{V}^\ell}] \otimes a \circ b^*) - \frac{1}{n}x \otimes [a, b^*],$$

$$[s \otimes b, s_n \otimes b^*] = \frac{1}{2\ell}([s, Id_{\mathcal{V}^\ell}] \otimes b \circ b^* + (s \circ Id_{\mathcal{V}^\ell}) \otimes [b, b^*]) - \frac{1}{n}s \otimes [b, b^*] + tr(ss_n)\langle b, b^* \rangle$$

and

$$[s_n \otimes b^*, v \otimes c] = \frac{1}{\ell}Id_{\mathcal{V}^\ell} v \otimes b^* \cdot c - \frac{1}{n}v \otimes b^* \cdot c.$$

We next note that

$$\begin{aligned}
 [\langle a_1, a_2 \rangle_n + \langle b_1, b_2 \rangle_n + \langle c_1, c_2 \rangle_n, x \otimes a] &= x \otimes (d_{a_1, a_2}^n + d_{b_1, b_2}^n + d_{c_1, c_2}^n)(a) \\
 &= \frac{1}{2n}x \otimes [b^*, a] \\
 [\langle a_1, a_2 \rangle_n + \langle b_1, b_2 \rangle_n + \langle c_1, c_2 \rangle_n, s \otimes b] &= s \otimes (d_{a_1, a_2}^n + d_{b_1, b_2}^n + d_{c_1, c_2}^n)(b) \\
 &= \frac{1}{2n}s \otimes [b^*, b] \\
 [\langle a_1, a_2 \rangle_n + \langle b_1, b_2 \rangle_n + \langle c_1, c_2 \rangle_n, v \otimes c] &= v \otimes (d_{a_1, a_2}^n + d_{b_1, b_2}^n + d_{c_1, c_2}^n)(c) \\
 &= \frac{1}{2n}v \otimes b^* \cdot c \\
 &- v \otimes \frac{1}{2}(f(c, c_2) \cdot c_1 + f(c, c_1) \cdot c_2).
 \end{aligned}$$

Therefore using Lemma 3.32, an easy verification gives that

$$\begin{aligned} [\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle, x \otimes a] &= -\frac{1}{4\ell}(x \circ Id_{\mathcal{V}^\ell} \otimes [a, b^*] + [x, Id_{\mathcal{V}^\ell}] \otimes a \circ b^*), \\ [\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle, s \otimes b] &= -\frac{1}{4\ell}([s, Id_{\mathcal{V}^\ell}] \otimes b \circ b^* + (s \circ Id_{\mathcal{V}^\ell}) \otimes [b, b^*]) \\ &\quad - \frac{1}{2\ell}tr(sId_{\mathcal{V}^\ell})\langle b, b^* \rangle, \end{aligned}$$

and

$$[\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle, v \otimes c] = \frac{1}{2\ell}Id_{\mathcal{V}^\ell} v \otimes b^* \cdot c - \frac{1}{2}v \otimes (f(c, c_2) \cdot c_1 + f(c, c_1) \cdot c_2).$$

These together with (3.23) complete the proof. \square

3.2.2. Types A and D. Suppose that I is an index set of cardinality $n + 1 > 5$ and I_0 is a subset of I of cardinality $\ell + 1 > 5$. Suppose that R is an irreducible finite root system of type $X = \dot{A}_I$ or D_I . Suppose that \mathcal{V} is a vector space with a basis $\{v_i \mid i \in I\}$ and take \mathcal{G} to be the finite dimensional split simple Lie algebra of type X as in Lemma 2.5 or Lemma 2.10 respectively and set $\mathcal{G}^\ell := \mathcal{G}_{I_0}$. Suppose that \mathcal{V}^ℓ is the subspace of \mathcal{V} spanned by $\{v_i \mid i \in I_0\}$. We take $Id_{\mathcal{V}}$ to be the identity map on \mathcal{V} and define $Id_{\mathcal{V}^\ell}$ as follows:

$$\begin{aligned} Id_{\mathcal{V}^\ell} : \mathcal{V} &\longrightarrow \mathcal{V} \\ v_i &\mapsto v_i, \quad v_j \mapsto 0; \quad (i \in I_0, j \in I \setminus I_0). \end{aligned}$$

Theorem 3.43. *Suppose that \mathcal{L} is a Lie algebra graded by the irreducible finite root system R of type $X = \dot{A}_I$ or D_I with grading pair $(\mathfrak{g}, \mathfrak{h})$ and let S be the irreducible full subsystem of R of type \dot{A}_{I_0} or D_{I_0} respectively.*

(i) *Consider \mathcal{L}^S as a \mathfrak{g}^S -module and take*

$$(3.44) \quad \mathcal{L}^S = \bigoplus_{i \in \mathcal{I}} \mathfrak{g}_i \oplus E$$

to be the decomposition of \mathcal{L}^S into finite dimensional irreducible \mathfrak{g}^S -submodules in which \mathcal{I} is an index set and for $i \in \mathcal{I}$, \mathfrak{g}_i is isomorphic to $\mathfrak{g}^S (\simeq \mathcal{G}^\ell)$, and E is a trivial \mathfrak{g}^S -submodule. Then there exists a class $\{\mathcal{D}_n, \mathcal{G}_i \mid i \in \mathcal{I}\}$ of finite dimensional \mathfrak{g} -submodules of \mathcal{L} such that

- \mathcal{D}_n is a trivial \mathfrak{g} -module and \mathcal{G}_i is isomorphic to $\mathfrak{g} \simeq \mathcal{G}$ for $i \in \mathcal{I}$,
- $\mathfrak{g}_i \subseteq \mathcal{G}_i$, ($i \in \mathcal{I}$),
- $\mathcal{L} = \bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \mathcal{D}_n$.

(ii) *Take \mathcal{A} to be a vector space with basis $\{a_i \mid i \in \mathcal{I}\}$ and identify \mathcal{L} with $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n$, say via the natural identification*

$$\varphi : \mathcal{L} \longrightarrow (\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n.$$

Transfer the Lie algebraic structure of \mathcal{L} to $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n$. Then $\mathcal{D}_\ell := \varphi(E)$ is a subalgebra of $\varphi(\mathcal{L}^S) = (\mathcal{G}^\ell \dot{\otimes} \mathcal{A}) \oplus \mathcal{D}_\ell$ and \mathcal{D}_n is a subalgebra of $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n$. Moreover the vector space \mathcal{A} is equipped with an associative algebraic structure if $X = \dot{A}_I$ and with a commutative associative algebraic structure if $X = D_I$.

(iii) There is a subspace \mathcal{K}_1 of the full skew-dihedral homology group of \mathcal{A} with respect to n and a subspace \mathcal{K}_2 of the full skew-dihedral homology group of \mathcal{A} with respect to ℓ such that \mathcal{D}_n and \mathcal{D}_ℓ are isomorphic to the quotient algebras $\{\mathcal{A}, \mathcal{A}\}_n / \mathcal{K}_1$ and $\{\mathcal{A}, \mathcal{A}\}_\ell / \mathcal{K}_2$ respectively, say via

$$\psi_1 : \{\mathcal{A}, \mathcal{A}\}_n / \mathcal{K}_1 \longrightarrow \mathcal{D}_n \quad \text{and} \quad \psi_2 : \{\mathcal{A}, \mathcal{A}\}_\ell / \mathcal{K}_2 \longrightarrow \mathcal{D}_\ell.$$

(iv) For $a, a' \in \mathcal{A}$, take

$$\langle a, a' \rangle_n := \psi_1(\{a, a'\}_n + \mathcal{K}_1) \quad \text{and} \quad \langle a, a' \rangle_\ell := \psi_2(\{a, a'\}_\ell + \mathcal{K}_2).$$

Then for $a, a' \in \mathcal{A}$, we have

$$\langle a, a' \rangle_n = \langle a, a' \rangle_\ell + \left(\left(\frac{1}{n+1} \text{Id}_{\mathcal{V}} - \frac{1}{\ell+1} \text{Id}_{\mathcal{V}^\ell} \right) \otimes (aa' - a'a) \right).$$

(v) For $a, a' \in \mathcal{A}$, set

$$\langle a, a' \rangle^n := \varphi^{-1}(\langle a, a' \rangle_n) \quad \text{and} \quad \langle a, a' \rangle^\ell := \varphi^{-1}(\langle a, a' \rangle_\ell).$$

If $\ell < n$, then for $a_1, a'_1, \dots, a_t, a'_t \in \mathcal{A}$, we have $\sum_{i=1}^t \langle a_i, a'_i \rangle^\ell = 0$ if and only if $\sum_{i=1}^t \langle a_i, a'_i \rangle^n = 0$ and $\sum_{i=1}^t [a_i, a'_i] = 0$.

(vi) For $x, y \in \mathcal{G}$, set $x \circ y := xy + yx - \frac{2\text{tr}(xy)}{\ell+1} \text{Id}_{\mathcal{V}^\ell}$ and for $a, a' \in \mathcal{A}$, set $\langle a, a' \rangle := \langle a, a' \rangle_\ell$. The Lie bracket on $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n = (\mathcal{G} \otimes \mathcal{A}) + \mathcal{D}_\ell$ is given by

(3.45)

$$\begin{aligned} [x \otimes a, y \otimes a'] &= \begin{cases} [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + \text{tr}(xy) \langle a, a' \rangle & X = \dot{A}_I, \\ [x, y] \otimes aa' + \text{tr}(xy) \langle a, a' \rangle & X = D_I, \end{cases} \\ [\langle a_1, a_2 \rangle, x \otimes a] &= \begin{cases} \frac{-1}{2(\ell+1)}(x \circ \text{Id}_{\mathcal{V}^\ell} \otimes [a, [a_1, a_2]]) \\ + [x, \text{Id}_{\mathcal{V}^\ell}] \otimes a \circ [a_1, a_2] + 2\text{tr}(\text{Id}_{\mathcal{V}^\ell} x) \langle a, [a_1, a_2] \rangle, & X = \dot{A}_I, \\ 0 & X = D_I, \end{cases} \\ [\langle a_1, a_2 \rangle, \langle a'_1, a'_2 \rangle] &= \begin{cases} \langle d_{a_1, a_2}^{\ell, \mathcal{A}}(a'_1), a'_2 \rangle + \langle a'_1, d_{a_1, a_2}^{\ell, \mathcal{A}}(a'_2) \rangle, & X = \dot{A}_I, \\ 0 & X = D_I, \end{cases} \end{aligned}$$

for $x, y \in \mathcal{G}$, $a, a', a_1, a_2, a'_1, a'_2 \in \mathcal{A}$.

3.2.3. Types B and C. Suppose that I is a nonempty index set of cardinality n greater than 4 and I_0 is a subset of I of cardinality $\ell > 4$. Take \mathcal{G} to be either $\mathfrak{o}_B(I)$ or $\mathfrak{sp}(I)$. Suppose that \mathcal{V} is a vector space with a basis $\{v_0, v_i, v_{\bar{i}} \mid i \in I\}$ equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) as in (2.8) if $\mathcal{G} = \mathfrak{o}_B(I)$ and it is a vector space with a basis $\{v_i, v_{\bar{i}} \mid i \in I\}$ equipped with a nondegenerate skew-symmetric bilinear form

(\cdot, \cdot) as in (2.13) if $\mathcal{G} := \mathfrak{sp}(I)$. Consider (2.21) and (2.23) and set $\mathcal{V}^\ell := \mathcal{V}_{I_0}$. Set

$$J := \begin{cases} I_0 \cup \bar{I}_0 \cup \{0\} & \text{if } \mathcal{G} = \mathfrak{o}_B(I) \\ I_0 \cup \bar{I}_0 & \text{if } \mathcal{G} = \mathfrak{sp}(I) \end{cases}$$

and define $Id_{\mathcal{V}^\ell} : \mathcal{V} \longrightarrow \mathcal{V}$ to be the linear transformation defined by

$$v_i \mapsto \begin{cases} v_i & \text{if } i \in J \\ 0 & \text{if } i \in I \cup \bar{I} \setminus J. \end{cases}$$

Finally set $\mathcal{S} := \mathcal{V}$ and $\mathcal{S}^\ell := \mathcal{V}^\ell$ if $\mathcal{G} := \mathfrak{o}_B(I)$ and take \mathcal{S} and $\mathcal{S}^\ell := \mathcal{S}_{I_0}$ to be as in (2.22) and (2.23) respectively if $\mathcal{G} = \mathfrak{sp}(I)$.

Theorem 3.46. *Suppose that \mathcal{L} is a Lie algebra graded by a root system R of type $X = B_I$ or C_I with grading pair $(\mathfrak{g}, \mathfrak{h})$ and let S be the irreducible full subsystem of R of type B_{I_0} or C_{I_0} respectively.*

(i) *Consider \mathcal{L}^S as a \mathfrak{g}^S -module and take*

$$(3.47) \quad \mathcal{L}^S = \bigoplus_{i \in \mathcal{I}} \mathfrak{g}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathfrak{s}_j \oplus E$$

to be the decomposition of \mathcal{L}^S into finite dimensional irreducible \mathfrak{g}^S -submodules in which \mathcal{I}, \mathcal{J} are index sets and for $i \in \mathcal{I}$ and $j \in \mathcal{J}$, \mathfrak{g}_i is isomorphic to $\mathfrak{g}^S (\simeq \mathcal{G}^\ell)$, \mathfrak{s}_j is isomorphic to \mathcal{S}^ℓ , and E is a trivial \mathfrak{g}^S -submodule. Then there exists a class $\{\mathcal{D}_n, \mathcal{G}_i, \mathcal{S}_j \mid i \in \mathcal{I}, j \in \mathcal{J}\}$ of finite dimensional \mathfrak{g} -submodules of \mathcal{L} such that

- \mathcal{D}_n is a trivial \mathfrak{g} -module, \mathcal{G}_i is isomorphic to $\mathfrak{g} (\simeq \mathcal{G})$ and \mathcal{S}_j is isomorphic to \mathcal{S} , for $i \in \mathcal{I}, j \in \mathcal{J}$,
- $\mathfrak{g}_i \subseteq \mathcal{G}_i, \mathfrak{s}_j \subseteq \mathcal{S}_j, (i \in \mathcal{I}, j \in \mathcal{J})$,
- $\mathcal{L} = \bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathcal{S}_j \oplus \mathcal{D}_n$.

(ii) *Take \mathcal{A} and \mathcal{B} to be vector spaces with bases $\{a_i \mid i \in \mathcal{I}\}$ and $\{b_j \mid j \in \mathcal{J}\}$ respectively and identify \mathcal{L} with $(\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{V} \otimes \mathcal{B}) \oplus \mathcal{D}_n$, say via the natural identification*

$$\varphi : \mathcal{L} \longrightarrow \bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathcal{S}_j \oplus \mathcal{D}_n.$$

Transfer the Lie algebraic structure of \mathcal{L} to $\bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathcal{S}_j \oplus \mathcal{D}_n$. Then $\mathcal{D}_\ell := \varphi(E)$ and \mathcal{D}_n are subalgebras of \mathcal{L} .

(iii) *Set $\mathfrak{a} := \mathcal{A} \oplus \mathcal{B}$. If $\mathcal{G} = \mathfrak{o}_B(I)$, \mathcal{A} is equipped with a unital commutative associative algebraic structure and the vector space \mathcal{B} is equipped with a unital \mathcal{A} -module structure. Also there is a symmetric \mathcal{A} -bilinear form $f : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{A}$ and $\mathfrak{a} = \mathcal{J}(f, \mathcal{B})$. Also if $\mathcal{G} = \mathfrak{sp}(I)$, \mathfrak{a} is equipped with a star algebraic structure with an involution $*$ such that \mathcal{A} (resp. \mathcal{B}) is the set of $*$ -fixed (resp. $*$ -skew fixed) points of \mathfrak{a} .*

(iv) There is a subspace \mathcal{K}_1 of the full skew-dihedral homology group of \mathfrak{a} with respect to n and a subspace \mathcal{K}_2 of the full skew-dihedral homology group of \mathfrak{a} with respect to ℓ such that \mathcal{D}_n and \mathcal{D}_ℓ are isomorphic to the quotient algebras $\{\mathfrak{a}, \mathfrak{a}\}_n/\mathcal{K}_1$ and $\{\mathfrak{a}, \mathfrak{a}\}_\ell/\mathcal{K}_2$ respectively, say via $\psi_1 : \{\mathfrak{a}, \mathfrak{a}\}_n/\mathcal{K}_1 \longrightarrow \mathcal{D}_n$ and $\psi_2 : \{\mathfrak{a}, \mathfrak{a}\}_\ell/\mathcal{K}_2 \longrightarrow \mathcal{D}_\ell$.

(v) For $\alpha, \alpha' \in \mathfrak{a}$, take

$$\langle \alpha, \alpha' \rangle_n := \{\alpha, \alpha'\}_n + \mathcal{K}_1 \quad \text{and} \quad \langle \alpha, \alpha' \rangle_\ell := \{\alpha, \alpha'\}_\ell + \mathcal{K}_2.$$

Then if $\alpha, \alpha' \in \mathcal{A}$ or $\alpha, \alpha' \in \mathcal{B}$, we have

$$\langle \alpha, \alpha' \rangle_n = \langle \alpha, \alpha' \rangle_\ell + \left(\left(\frac{1}{n} Id_{\mathcal{V}} - \frac{1}{\ell} Id_{\mathcal{V}^\ell} \right) \otimes (1/2)(\alpha\alpha' - \alpha'\alpha) \right).$$

(vi) For $\alpha, \alpha' \in \mathfrak{a}$, set

$$\langle \alpha, \alpha' \rangle^n := \varphi^{-1}(\langle \alpha, \alpha' \rangle_n) \quad \text{and} \quad \langle \alpha, \alpha' \rangle^\ell := \varphi^{-1}(\langle \alpha, \alpha' \rangle_\ell).$$

If $\ell < n$, then for $a_1, a'_1, \dots, a_t, a'_t \in \mathcal{A}$ and $b_1, b'_1, \dots, b_t, b'_t \in \mathcal{B}$, we have $\sum_{i=1}^t \langle a_i, a'_i \rangle^\ell + \sum_{i=1}^t \langle b_i, b'_i \rangle^\ell = 0$ if and only if

$$\sum_{i=1}^t \langle a_i, a'_i \rangle^n + \sum_{i=1}^t \langle b_i, b'_i \rangle^n = 0 \quad \text{and} \quad \sum_{i=1}^t [a_i, a'_i] + \sum_{i=1}^t [b_i, b'_i] = 0.$$

(vii) For $e, f \in \mathcal{G} \cup \mathcal{S}$, set

$$e \circ f := ef + fe - \frac{tr(ef)}{\ell} Id_{\mathcal{V}^\ell},$$

and for $\alpha, \alpha' \in \mathfrak{a}$, set

$$\langle \alpha, \alpha' \rangle := \langle \alpha, \alpha' \rangle_\ell,$$

the Lie bracket on $(\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus \mathcal{D}_n = ((\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B})) + \mathcal{D}_\ell$ is given by

$$\begin{aligned} [x \otimes a, y \otimes a'] &= [x, y] \otimes aa' + tr(xy) \langle a, a' \rangle, \\ [x \otimes a, s \otimes b] &= xs \otimes ab, \\ [s \otimes b, t \otimes b'] &= D_{s,t} \otimes f(b, b') + (s, t) \langle b, b' \rangle, \\ (3.48) \quad [\langle \alpha, \alpha' \rangle, x \otimes a] &= x \otimes d_{\alpha, \alpha'}^{\ell, \mathfrak{a}}(a), \\ [\langle \alpha, \alpha' \rangle, s \otimes b] &= s \otimes d_{\alpha, \alpha'}^{\ell, \mathfrak{a}}(b), \\ [\langle \alpha_1, \alpha_2 \rangle, \langle \alpha'_1, \alpha'_2 \rangle] &= \langle d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(\alpha'_1), \alpha'_2 \rangle + \langle \alpha'_1, d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(\alpha'_2) \rangle. \end{aligned}$$

(see Definition 2.6) for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, and $\alpha, \alpha', \alpha_1, \alpha_2, \alpha'_1, \alpha'_2 \in \mathfrak{a}$ if $\mathcal{G} = \mathfrak{o}_B(I)$ and it is given by

$$\begin{aligned} (3.49) \quad [x \otimes a, y \otimes a'] &= [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + tr(xy) \langle a, a' \rangle, \\ [x \otimes a, s \otimes b] &= (x \circ s) \otimes \frac{1}{2}[a, b] + [x, s] \otimes \frac{1}{2}(a \circ b), \\ [s \otimes b, t \otimes b'] &= [s, t] \otimes \frac{1}{2}(b \circ b') + (s \circ t) \otimes \frac{1}{2}[b, b'] + tr(st) \langle b, b' \rangle, \\ [\langle \alpha, \alpha' \rangle, x \otimes a] &= \frac{-1}{4\ell}((x \circ Id_{\mathcal{V}^\ell}) \otimes [a, \beta_{\alpha, \alpha'}^*] + [x, Id_{\mathcal{V}^\ell}] \otimes (a \circ \beta_{\alpha, \alpha'}^*)), \\ [\langle \alpha, \alpha' \rangle, s \otimes b] &= \frac{-1}{4\ell}([s, Id_{\mathcal{V}^\ell}] \otimes (b \circ \beta_{\alpha, \alpha'}^*) + (s \circ Id_{\mathcal{V}^\ell}) \otimes [b, \beta_{\alpha, \alpha'}^*] + 2tr(s Id_{\mathcal{V}^\ell}) \langle b, \beta_{\alpha, \alpha'}^* \rangle), \\ [\langle \alpha_1, \alpha_2 \rangle, \langle \alpha'_1, \alpha'_2 \rangle] &= \langle d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(\alpha'_1), \alpha'_2 \rangle + \langle \alpha'_1, d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(\alpha'_2) \rangle. \end{aligned}$$

(see (3.7)) for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $\alpha, \alpha', \alpha_1, \alpha_2, \alpha'_1, \alpha'_2 \in \mathfrak{a}$ if $\mathcal{G} = \mathfrak{sp}(I)$.

4. ROOT GRADED LIE ALGEBRAS - GENERAL CASE

In this section, we discuss certain recognition theorems to characterize Lie algebras graded by an infinite irreducible locally finite root system. The main target of the present section is to generalize the decomposition (3.1) for Lie algebras graded by infinite root systems. For a Lie algebra \mathcal{L} graded by an infinite locally finite root system with grading pair $(\mathfrak{g}, \mathfrak{h})$, we first decompose \mathcal{L} as a direct sum of a certain subalgebra of \mathcal{L} and a certain locally finite completely reducible \mathfrak{g} -submodule. This in particular results in a generalized decomposition for \mathcal{L} as in (3.1). We next reconstruct the structure of \mathcal{L} in terms of the ingredients involved in this decomposition. Moreover we prove that any Lie algebra graded by an irreducible locally finite root system arises in this way. As in the previous section, we concentrate our attention on type BC and for other types, we just report the results.

4.1. Recognition theorem for type BC . Suppose that I is an infinite index set and ℓ is a positive integer greater than 3. We assume R is an irreducible locally finite root system of type BC_I and take \mathcal{G} , \mathcal{S} and \mathcal{V} to be as in Lemmas 2.12 and 2.20. We show that an R -graded Lie algebra \mathcal{L} can be decomposed into

$$(4.1) \quad (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D}$$

in which \mathcal{A}, \mathcal{B} and \mathcal{C} are vector spaces and \mathcal{D} is a subalgebra of \mathcal{L} . We equip $\mathfrak{b} := \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$ with a unital associative star algebraic structure and show that \mathcal{D} can be expressed as a quotient of the algebra $\{\mathfrak{b}, \mathfrak{b}\}_\ell$ by a subspace of the full skew-dihedral homology group of \mathfrak{b} with respect to ℓ satisfying the uniform property on \mathfrak{b} . Conversely, for vector spaces A, B, C and D with specific natures, we form the decomposition (4.1), equip it with a Lie bracket and show that it is an R -graded Lie algebra.

Theorem 4.2. *Suppose that I is an infinite index set and ℓ is an integer greater than 3. Assume R is an irreducible locally finite root system of type BC_I and \mathcal{V} is a vector space with a basis $\{v_i \mid i \in I \cup \bar{I}\}$. Suppose that (\cdot, \cdot) is a bilinear form as in (2.13), set $\mathcal{G} := \mathfrak{sp}(I)$ and consider \mathcal{S} as in (2.22). Fix a subset I_0 of I of cardinality ℓ and take R_0 to be the full irreducible subsystem of R of type BC_{I_0} . Suppose that $\{R_\lambda \mid \lambda \in \Lambda\}$ is the class of all finite irreducible full subsystems of R containing R_0 , where Λ is an index set containing zero. For $\lambda \in \Lambda$, take \mathcal{G}^λ as in Lemma 2.14 and $\mathcal{V}^\lambda, \mathcal{S}^\lambda$ as in (2.23), also define*

$$\mathfrak{I}_\lambda : \mathcal{V} \longrightarrow \mathcal{V} \\ v_i \mapsto \begin{cases} v_i & i \in I_\lambda \cup \bar{I}_\lambda \\ 0 & \text{otherwise.} \end{cases}$$

For $e, f \in \mathcal{G} \cup \mathcal{S}$, define

$$e \circ f := ef + fe - \frac{\text{tr}(ef)}{l} \mathfrak{I}_0.$$

(i) Suppose that $(\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple of type BC and \mathcal{A}, \mathcal{B} are $*$ -fixed and $*$ -skew fixed points of \mathfrak{a} respectively. Set $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ and take $[\cdot, \cdot], \circ, \heartsuit, \diamond$ to be as in Subsection 3.1. For $\beta_1, \beta_2 \in \mathfrak{b}$, consider $d_{\beta_1, \beta_2}^{\ell, \mathfrak{b}}$ as in (3.5) and take $\beta_{\beta_1, \beta_2}^*, \beta_1^*$ and β_2^* as in Proposition 3.40. For a subset \mathcal{K} of $FH(\mathfrak{b})$ satisfying the uniform property on \mathfrak{b} , set

$$\mathcal{L}(\mathfrak{b}, \mathcal{K}) := (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus (\{\mathfrak{b}, \mathfrak{b}\}_\ell / \mathcal{K}).$$

Then setting $\langle \beta, \beta' \rangle := \{\beta, \beta'\} + \mathcal{K}$, $\beta, \beta' \in \mathfrak{b}$, $\mathcal{L}(\mathfrak{b}, \mathcal{K})$ together with (4.3)

$$\begin{aligned} [x \otimes a, y \otimes a'] &= [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + \text{tr}(xy) \langle a, a' \rangle, \\ [x \otimes a, s \otimes b] &= (x \circ s) \otimes \frac{1}{2}[a, b] + [x, s] \otimes \frac{1}{2}(a \circ b) = -[s \otimes b, x \otimes a], \\ [s \otimes b, t \otimes b'] &= [s, t] \otimes \frac{1}{2}(b \circ b') + (s \circ t) \otimes \frac{1}{2}[b, b'] + \text{tr}(st) \langle b, b' \rangle, \\ [x \otimes a, u \otimes c] &= xu \otimes a \cdot c = -[u \otimes c, x \otimes a], \\ [s \otimes b, u \otimes c] &= su \otimes b \cdot c = -[u \otimes c, s \otimes b], \\ [u \otimes c, v \otimes c'] &= (u \circ v) \otimes (c \diamond c') + [u, v] \otimes (c \heartsuit c') + (u, v) \langle c, c' \rangle, \\ [\langle \beta_1, \beta_2 \rangle, x \otimes a] &= \frac{-1}{4\ell} ((x \circ \mathfrak{I}_0) \otimes [a, \beta_{\beta_1, \beta_2}^*] + [x, \mathfrak{I}_0] \otimes (a \circ \beta_{\beta_1, \beta_2}^*)), \\ [\langle \beta_1, \beta_2 \rangle, s \otimes b] &= \frac{-1}{4\ell} ([s, \mathfrak{I}_0] \otimes (b \circ \beta_{\beta_1, \beta_2}^*) + (s \circ \mathfrak{I}_0) \otimes [b, \beta_{\beta_1, \beta_2}^*] + 2\text{tr}(s\mathfrak{I}_0) \langle b, \beta_{\beta_1, \beta_2}^* \rangle), \\ [\langle \beta_1, \beta_2 \rangle, v \otimes c] &= \frac{1}{2\ell} \mathfrak{I}_0 v \otimes \beta_{\beta_1, \beta_2}^* \cdot c - \frac{1}{2} v \otimes (f(c, \beta_2^*) \cdot \beta_1^* + f(c, \beta_1^*) \cdot \beta_2^*) \\ [\langle \beta_1, \beta_2 \rangle, \langle \beta'_1, \beta'_2 \rangle] &= \langle d_{\beta_1, \beta_2}^{\ell}(\beta'_1), \beta'_2 \rangle + \langle \beta'_1, d_{\beta_1, \beta_2}^{\ell}(\beta'_2) \rangle \end{aligned}$$

for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $u, v \in \mathcal{V}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $c, c' \in \mathcal{C}$, $\beta_1, \beta_2, \beta'_1, \beta'_2 \in \mathfrak{b}$, is an R -graded Lie algebra with grading pair $(\mathcal{G}, \mathcal{H})$ where \mathcal{H} is the splitting Cartan subalgebra of \mathcal{G} defined in Lemma 2.12.

(ii) If \mathcal{L} is an R -graded Lie algebra with grading pair $(\mathfrak{g}, \mathfrak{h})$, then there is a coordinate quadruple $(\mathfrak{a}, *, \mathcal{C}, f)$ of type BC and a subspace \mathcal{K} of $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ satisfying the uniform property on \mathfrak{b} such that \mathcal{L} is isomorphic to $\mathcal{L}(\mathfrak{b}, \mathcal{K})$.

Proof. (i) We prove that $\mathcal{L}(\mathfrak{b}, \mathcal{K})$ together with (4.3) is a Lie algebra. For $\lambda \in \Lambda$ set $n_\lambda := |I_\lambda|$ and $\mathcal{L}^\lambda := (\mathcal{G}^\lambda \dot{\otimes} \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot{\otimes} \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot{\otimes} \mathcal{C}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle$. Also for $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, and $c, c' \in \mathcal{C}$, set

$$\begin{aligned} \langle a, a' \rangle_\lambda &:= (((\frac{-1}{\ell} \mathfrak{I}_0 + \frac{1}{n_\lambda} \mathfrak{I}_\lambda) \otimes \frac{1}{2}[a, a']) + \langle a, a' \rangle), \\ \langle b, b' \rangle_\lambda &:= (((\frac{-1}{\ell} \mathfrak{I}_0 + \frac{1}{n_\lambda} \mathfrak{I}_\lambda) \otimes \frac{1}{2}[b, b']) + \langle b, b' \rangle), \\ \langle c, c' \rangle_\lambda &:= (((\frac{1}{\ell} \mathfrak{I}_0 - \frac{1}{n_\lambda} \mathfrak{I}_\lambda) \otimes \frac{1}{2}c \heartsuit c') + \langle c, c' \rangle), \\ \langle a, b \rangle_\lambda &= \langle b, c \rangle_\lambda = \langle a, c \rangle_\lambda := 0. \end{aligned}$$

Take $\langle \mathfrak{b}, \mathfrak{b} \rangle_\lambda := \text{span}\{\langle a, a' \rangle_\lambda, \langle b, b' \rangle_\lambda, \langle c, c' \rangle_\lambda \mid a, a' \in \mathcal{A}, b, b' \in \mathcal{B}, c, c' \in \mathcal{C}\}$ and note that as \mathcal{K} satisfies the uniform property on \mathfrak{b} , we have

$$\mathcal{L}^\lambda := (\mathcal{G}^\lambda \dot{\otimes} \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot{\otimes} \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot{\otimes} \mathcal{C}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle_\lambda.$$

For $\beta = a + b + c$, $\beta' = a' + b' + c' \in \mathfrak{b}$, set $\langle \beta, \beta' \rangle_\lambda := \langle a, a' \rangle_\lambda + \langle b, b' \rangle_\lambda + \langle c, c' \rangle_\lambda$.

Now consider the linear transformation $\psi : \mathfrak{b} \otimes \mathfrak{b} \longrightarrow \langle \mathfrak{b}, \mathfrak{b} \rangle_\lambda$ mapping $\beta \otimes \beta'$ to $\langle \beta, \beta' \rangle_\lambda$. It is not difficult to see that for the subspace K of $\mathfrak{b} \otimes \mathfrak{b}$ defined in Subsection 3.1, $\psi(K) = \{0\}$. So ψ induces a linear transformation $\dot{\psi} : \{\mathfrak{b}, \mathfrak{b}\}_{n_\lambda} \longrightarrow \langle \mathfrak{b}, \mathfrak{b} \rangle_\lambda$ mapping $\{\beta, \beta'\}_{n_\lambda}$ to $\langle \beta, \beta' \rangle_\lambda$. Take \mathcal{K}_λ to be the kernel of $\dot{\psi}$. If $t \in \mathbb{N}$, $a_i, a'_i \in \mathcal{A}$, $b_i, b'_i \in \mathcal{B}$ and $c_i, c'_i \in \mathcal{C}$ ($1 \leq i \leq t$) are such that $\sum_{i=1}^t (\{a_i, a'_i\}_{n_\lambda} + \{b_i, b'_i\}_{n_\lambda} + \{c_i, c'_i\}_{n_\lambda}) \in \mathcal{K}_{n_\lambda}$, then $\sum_{i=1}^t (\langle a_i, a'_i \rangle_\lambda + \langle b_i, b'_i \rangle_\lambda + \langle c_i, c'_i \rangle_\lambda) = 0$. This implies that

$$(4.4) \quad \left(\left(\frac{-1}{\ell} \mathfrak{I}_0 + \frac{1}{n_\lambda} \mathfrak{I}_\lambda \right) \otimes \frac{1}{2} \sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - (c_i \heartsuit c'_i)) + \sum_{i=1}^t (\langle a_i, a'_i \rangle + \langle b_i, b'_i \rangle + \langle c_i, c'_i \rangle) \right) = 0.$$

This in turn implies that $\sum_{i=1}^t (\langle a_i, a'_i \rangle + \langle b_i, b'_i \rangle + \langle c_i, c'_i \rangle) = 0$. Therefore we get that \mathcal{K}_λ is a subset of the full skew-dihedral homology group of \mathfrak{b} with respect to ℓ . But if $\lambda \neq 0$, (4.4) implies that $\sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - (c_i \heartsuit c'_i)) = 0$. Now one gets using this together with the fact that \mathcal{K}_λ is a subset of the full skew-dihedral homology group of \mathfrak{b} with respect to ℓ , that \mathcal{K}_λ is a subset of the full skew-dihedral homology group of \mathfrak{b} with respect to n_λ . Now it follows from [3, Chapter V] that \mathcal{L}^λ together with the product introduced in (4.3) restricted to $\mathcal{L}^\lambda \times \mathcal{L}^\lambda$ defines a Lie algebra. Therefore \mathcal{L} together with $[\cdot, \cdot]$ is a Lie algebra as $\mathcal{L} = \cup_{\lambda \in \Lambda} \mathcal{L}^\lambda$. Now one can easily see that \mathcal{L} has a weight space decomposition $\mathcal{L} = \oplus_{\alpha \in R} \mathcal{L}_\alpha$ with respect to \mathcal{H} in which

$$\mathcal{L}_\alpha = \begin{cases} \mathcal{V}_\alpha \otimes \mathcal{C} & \text{if } \alpha \in R_{sh} \\ (\mathcal{G}_\alpha \otimes \mathcal{A}) \oplus (\mathcal{S}_\alpha \otimes \mathcal{B}) & \text{if } \alpha \in R_{lg} \\ \mathcal{G}_\alpha \otimes \mathcal{A} & \text{if } \alpha \in R_{ex} \\ (\mathcal{G}_0 \otimes \mathcal{A}) \oplus (\mathcal{S}_0 \otimes \mathcal{B}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle & \text{if } \alpha = 0 \end{cases}$$

and that \mathcal{L} is an R -graded Lie algebra with grading pair $(\mathcal{G}, \mathcal{H})$.

(ii) For $\lambda \in \Lambda$, set

$$\mathcal{L}^\lambda := \sum_{\alpha \in R_\lambda^\times} \mathcal{L}_\alpha \oplus \sum_{\alpha \in R_\lambda^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}],$$

$$\mathfrak{g}^\lambda := \sum_{\alpha \in (R_\lambda)_{sdiv}^\times} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in (R_\lambda)_{sdiv}^\times} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

and note that \mathfrak{g}^λ is isomorphic to \mathcal{G}^λ . We know by Lemma 3.10 that \mathcal{L}^λ is an R_λ -graded Lie algebra with grading pair $(\mathfrak{g}^\lambda, \mathfrak{h}^\lambda := \mathfrak{g}^\lambda \cap \mathfrak{h})$. Consider \mathcal{L}^0 as a \mathfrak{g}^0 -module and suppose that $\{\mathcal{G}_i^0, \mathcal{S}_j^0, \mathcal{V}_t^0, \mathcal{D}_0 \mid i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T}\}$ is a class of finite dimensional \mathfrak{g}^0 -submodules of \mathcal{L}^0 such that

- $\mathcal{L}^0 = \sum_{i \in \mathcal{I}} \mathcal{G}_i^0 \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^0 \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^0 \oplus \mathcal{D}_0$,
- \mathcal{D}_0 is a trivial \mathfrak{g}^0 -submodule of \mathcal{L}^0 ,
- for $i \in \mathcal{I}, j \in \mathcal{J}$ and $t \in \mathcal{T}$, \mathcal{G}_i^0 is isomorphic to \mathcal{G}^0 , \mathcal{S}_j^0 is isomorphic to \mathcal{S}^0 , and \mathcal{V}_t^0 is isomorphic to \mathcal{V}^0 .

Now for $\lambda \in \Lambda$, consider \mathcal{L}^λ as a \mathfrak{g}^λ -module via the adjoint representation. Using Lemmas 3.16 and 3.24, one finds finite dimensional irreducible \mathfrak{g}^λ -submodules $\mathcal{G}_i^\lambda, \mathcal{S}_j^\lambda, \mathcal{V}_t^\lambda$ ($i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T}$) of \mathcal{L}^λ and a trivial

\mathfrak{g}^λ -submodule \mathcal{D}_λ such that

$$(\mathcal{I}, \mathcal{J}, \mathcal{T}, \{\mathcal{G}_i^0\}, \{\mathcal{G}_i^\lambda\}, \{\mathcal{S}_j^0\}, \{\mathcal{S}_j^\lambda\}, \{\mathcal{V}_t^0\}, \{\mathcal{V}_t^\lambda\}, \mathcal{D}_0, \mathcal{D}_\lambda)$$

is an (R^λ, R^0) -datum for the pair $(\mathcal{L}^\lambda, \mathcal{L}^0)$ (see (3.17)). we know from Subsection 3.2 that there is a coordinate quadruple $(\mathfrak{a}, *, \mathcal{C}, f)$ of type BC and a subspace \mathcal{K}_λ of the full skew-dihedral homology group of $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ with respect to $n_\lambda = |I_\lambda|$ such that \mathcal{D}_λ is a subalgebra of \mathcal{L}^λ isomorphic to the quotient algebra $\{\mathfrak{b}, \mathfrak{b}\}_{n_\lambda}/\mathcal{K}_\lambda$, say via $\phi_\lambda : \{\mathfrak{b}, \mathfrak{b}\}_{n_\lambda}/\mathcal{K}_\lambda \longrightarrow \mathcal{D}_\lambda$. Now for $\beta, \beta' \in \mathfrak{b}$, set

$$(4.5) \quad \langle \beta, \beta' \rangle^\lambda := \phi_\lambda(\{\beta, \beta'\}_\lambda + \mathcal{K}_\lambda).$$

Take \mathcal{A} and \mathcal{B} to be the $*$ -fixed and $*$ -skew fixed points of \mathfrak{a} respectively and note that

$$\mathcal{D}_\lambda = \text{span}\{\langle a, a' \rangle^\lambda, \langle b, b' \rangle^\lambda, \langle c, c' \rangle^\lambda \mid a, a' \in \mathcal{A}, b, b' \in \mathcal{B}, c, c' \in \mathcal{C}\}.$$

We now proceed with the proof in the following steps:

Step 1: For $i \in \mathcal{I}$, $j \in \mathcal{J}$, $t \in \mathcal{T}$ and $\lambda, \mu \in \Lambda$ with $\lambda \preccurlyeq \mu$, \mathcal{G}_i^μ is the \mathfrak{g}^μ -submodule of \mathcal{L}^μ generated by \mathcal{G}_i^λ , \mathcal{S}_j^μ is the \mathfrak{g}^μ -submodule of \mathcal{L}^μ generated by \mathcal{S}_j^λ and \mathcal{V}_t^μ is the \mathfrak{g}^μ -submodule of \mathcal{L}^μ generated by \mathcal{V}_t^λ . In other words,

$$(\mathcal{I}, \mathcal{J}, \mathcal{T}, \{\mathcal{G}_i^\lambda\}, \{\mathcal{G}_i^\mu\}, \{\mathcal{S}_j^\lambda\}, \{\mathcal{S}_j^\mu\}, \{\mathcal{V}_t^\lambda\}, \{\mathcal{V}_t^\mu\}, \mathcal{D}_\lambda, \mathcal{D}_\mu)$$

is an (R^μ, R^λ) -datum for the pair $(\mathcal{L}^\mu, \mathcal{L}^\lambda)$: It is immediate using the facts that \mathfrak{g}^λ is a subalgebra of \mathfrak{g}^μ .

Step 2: For $\lambda \in \Lambda$, $\mathcal{L}^\lambda = \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda \oplus \mathcal{D}_0$: By Step 1 and Remark 3.39, $\mathcal{L}^\lambda = (\sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda) + \mathcal{D}_0$. Suppose $d \in \mathcal{D}_0$, $x \in \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda$ and $x + d = 0$. Since $d \in \mathcal{D}_0$, there are $t \in \mathbb{N}$, $a_i, a'_i \in \mathcal{A}$, $b_i, b'_i \in \mathcal{B}$ and $c_i, c'_i \in \mathcal{C}$ ($1 \leq i \leq t$) such that $d = \sum_{i=1}^t \langle a_i, a'_i \rangle^0 + \langle b_i, b'_i \rangle^0 + \langle c_i, c'_i \rangle^0$. It follows from Step 1 and Lemma 3.32 that there is $y \in \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda$ such that $d = y + \sum_{i=1}^t \langle a_i, a'_i \rangle^\lambda + \langle b_i, b'_i \rangle^\lambda + \langle c_i, c'_i \rangle^\lambda$. Now as $0 = x + d = x + y + \sum_{i=1}^t \langle a_i, a'_i \rangle^\lambda + \langle b_i, b'_i \rangle^\lambda + \langle c_i, c'_i \rangle^\lambda$, we get that $x + y = 0$ and $\sum_{i=1}^t (\langle a_i, a'_i \rangle^\lambda + \langle b_i, b'_i \rangle^\lambda + \langle c_i, c'_i \rangle^\lambda) = 0$. Take $\mu \in \Lambda$ to be such that $\lambda \preccurlyeq \mu$, then using Step 1, one gets that the pairs $(\mathcal{L}^\lambda, \mathcal{L}^\mu)$ and $(\mathcal{L}^0, \mathcal{L}^\lambda)$ play the same role as the pair $(\mathcal{L}^\ell, \mathcal{L}^n)$ in Subsection 3.2. Using Remark 3.39 for the pair $(\mathcal{L}^\lambda, \mathcal{L}^\mu)$, one gets that $\sum_{i=1}^t (\langle a_i, a'_i \rangle^\mu + \langle b_i, b'_i \rangle^\mu + \langle c_i, c'_i \rangle^\mu) = 0$ and $\sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - c_i \circ c'_i) = 0$. Next using Remark 3.39 for the pair $(\mathcal{L}^0, \mathcal{L}^\mu)$, we get that $d = \sum_{i=1}^t (\langle a_i, a'_i \rangle^0 + \langle b_i, b'_i \rangle^0 + \langle c_i, c'_i \rangle^0) = 0$.

Step 3: \mathcal{K}_0 satisfies the uniform property on \mathfrak{b} : Suppose that

$$\sum_{i=1}^t (\{a_i, a'_i\}_\ell + \{b_i, b'_i\}_\ell + \{c_i, c'_i\}_\ell) \in \mathcal{K}_0,$$

for $a_1, a'_1, \dots, a_t, a'_t \in \mathcal{A}$, $b_1, b'_1, \dots, b_t, b'_t \in \mathcal{B}$, and $c_1, c'_1, \dots, c_n, c'_n \in \mathcal{C}$, so $\sum_{i=1}^n (\langle a_i, a'_i \rangle^0 + \langle b_i, b'_i \rangle^0 + \langle c_i, c'_i \rangle^0) = 0$. Now take $\lambda \in \Lambda \setminus \{0\}$, then by Step 1, $(\mathcal{L}^0, \mathcal{L}^\lambda)$ plays the same role as the pair $(\mathcal{L}^\ell, \mathcal{L}^n)$ in Subsection 3.2 and so an argument analogous to the proof of Step 2 shows that $\sum_{i=1}^t ([a_i, a'_i] + [b_i, b'_i] - c_i \heartsuit c'_i) = 0$. This completes the proof.

Step 4:

$$\bigcup_{\lambda \in \Lambda} \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda = \sum_{i \in \mathcal{I}} \bigcup_{\lambda \in \Lambda} \mathcal{G}_i^\lambda, \bigcup_{\lambda \in \Lambda} \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda = \sum_{j \in \mathcal{J}} \bigcup_{\lambda \in \Lambda} \mathcal{S}_j^\lambda, \bigcup_{\lambda \in \Lambda} \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda = \sum_{t \in \mathcal{T}} \bigcup_{\lambda \in \Lambda} \mathcal{V}_t^\lambda :$$

We just prove the first equality; the other ones may be proved using an analogous argument. Suppose that $x \in \sum_{i \in \mathcal{I}} \bigcup_{\lambda \in \Lambda} \mathcal{G}_i^\lambda$, then there are $i_1, \dots, i_n \in \mathcal{I}$, $\mu_1, \dots, \mu_n \in \Lambda$ and $x_1 \in \mathcal{G}_{i_1}^{\mu_1}, \dots, x_n \in \mathcal{G}_{i_n}^{\mu_n}$ with $x = x_1 + \dots + x_n$. Take $\mu \in \Lambda$ to be such that $\mu_t \preccurlyeq \mu$ for all $1 \leq t \leq n$. Then by Step 1, for $1 \leq t \leq n$, $\mathcal{G}_{i_t}^{\mu_t} \subseteq \mathcal{G}_{i_t}^\mu$. Therefore $x = x_1 + \dots + x_n \in \sum_{i \in \mathcal{I}} \mathcal{G}_i^\mu \subseteq \bigcup_{\lambda \in \Lambda} \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda$. This completes the proof.

Step 5: We have

$$\bigcup_{\lambda \in \Lambda} \mathcal{L}^\lambda = \bigcup_{\lambda \in \Lambda} \left(\sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \right) + \bigcup_{\lambda \in \Lambda} \left(\sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \right) + \bigcup_{\lambda \in \Lambda} \sum_{t \in \mathcal{T}} (\mathcal{V}_t^\lambda) + \mathcal{D}_0 :$$

Using Step 2, one gets that the left hand side of the above equality is a subset of the right hand side. So we need to show the other side inequality. Suppose a is an element of the right hand side, then there are μ_1, μ_2, μ_3 , $x \in \sum_{i \in \mathcal{I}} \mathcal{G}_i^{\mu_1} \subseteq \mathcal{L}^{\mu_1}$, $y \in \sum_{j \in \mathcal{J}} \mathcal{S}_j^{\mu_2} \subseteq \mathcal{L}^{\mu_2}$, $z \in \sum_{t \in \mathcal{T}} \mathcal{V}_t^{\mu_3} \subseteq \mathcal{L}^{\mu_3}$ and $d \in \mathcal{D}_0 \subseteq \mathcal{L}^0$ such that $a = x + y + z + d$. Fix an upper bound $\mu \in \Lambda$ for $\{0, \mu_1, \mu_2, \mu_3\}$, then we have $a = x + y + z + d \in \mathcal{L}^\mu \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{L}^\lambda$.

Step 6: $(\sum_{i \in \mathcal{I}} \bigcup_{\lambda \in \Lambda} \mathcal{G}_i^\lambda) + (\sum_{j \in \mathcal{J}} \bigcup_{\lambda \in \Lambda} \mathcal{S}_j^\lambda) + (\sum_{t \in \mathcal{T}} \bigcup_{\lambda \in \Lambda} \mathcal{V}_t^\lambda) + \mathcal{D}_0$ is a direct sum: Suppose that $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m, \eta_1, \dots, \eta_p \in \Lambda$ and $i_1, \dots, i_n \in \mathcal{I}$, $j_1, \dots, j_m \in \mathcal{J}$ and $t_1, \dots, t_p \in \mathcal{T}$ are distinct. Let $x_{i_1} \in \mathcal{G}_{i_1}^{\lambda_1}, \dots, x_{i_n} \in \mathcal{G}_{i_n}^{\lambda_n}$, $y_{j_1} \in \mathcal{S}_{j_1}^{\mu_1}, \dots, y_{j_m} \in \mathcal{S}_{j_m}^{\mu_m}$, $z_{t_1} \in \mathcal{V}_{t_1}^{\eta_1}, \dots, z_{t_p} \in \mathcal{V}_{t_p}^{\eta_p}$ and $d \in \mathcal{D}_0$ be such that

$$x_{i_1} + \dots + x_{i_n} + y_{j_1} + \dots + y_{j_m} + z_{t_1} + \dots + z_{t_p} + d = 0.$$

Now take $\lambda \in \Lambda$ to be an upper bound for $\{\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m, \eta_1, \dots, \eta_p\}$ with respect to the partial ordering on Λ , then we get using Steps 1,2 that $x_{i_1} + \dots + x_{i_n} + y_{j_1} + \dots + y_{j_m} + z_{t_1} + \dots + z_{t_p} + d$ is a summation in $(\oplus_{i \in \mathcal{I}} \mathcal{G}_i^\lambda) \oplus (\oplus_{j \in \mathcal{J}} \mathcal{S}_j^\lambda) \oplus (\oplus_{t \in \mathcal{T}} \mathcal{V}_t^\lambda) \oplus \mathcal{D}_0$. Therefore we have

$$x_{i_1} = 0, \dots, x_{i_n} = 0, y_{j_1} = 0, \dots, y_{j_m} = 0, z_{t_1} = 0, \dots, z_{t_p} = 0, d = 0.$$

This completes the proof of this step.

Step 7: The assertion stated in (ii) is true: Take \mathcal{A} to be a vector space with a basis $\{a_i \mid i \in \mathcal{I}\}$, \mathcal{B} to be a vector space with a basis $\{b_j \mid j \in \mathcal{J}\}$,

and \mathcal{C} to be a vector space with a basis $\{c_t \mid t \in \mathcal{T}\}$. Using Steps 1-2,4-6, we get that

$$\begin{aligned} \mathcal{L} = \bigcup_{\lambda \in \Lambda} \mathcal{L}_\lambda &= \bigcup_{\lambda \in \Lambda} (\sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda) + \bigcup_{\lambda \in \Lambda} (\sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda) + \bigcup_{\lambda \in \Lambda} (\sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda) + \mathcal{D}_0 \\ &= (\bigoplus_{i \in \mathcal{I}} \bigcup_{\lambda \in \Lambda} \mathcal{G}_i^\lambda) \oplus (\bigoplus_{j \in \mathcal{J}} \bigcup_{\lambda \in \Lambda} \mathcal{S}_j^\lambda) \oplus (\bigoplus_{t \in \mathcal{T}} \bigcup_{\lambda \in \Lambda} \mathcal{V}_t^\lambda) \oplus \mathcal{D}_0. \end{aligned}$$

Now consider \mathcal{L} as a \mathfrak{g} -module via the adjoint representation and for $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $t \in \mathcal{T}$, set

$$\mathcal{G}^{(i)} := \bigcup_{\lambda \in \Lambda} \mathcal{G}_i^\lambda, \quad \mathcal{S}^{(j)} := \bigcup_{\lambda \in \Lambda} \mathcal{S}_j^\lambda, \quad \mathcal{V}^{(t)} := \bigcup_{\lambda \in \Lambda} \mathcal{V}_t^\lambda,$$

then by Propositions 2.19 and 2.20, $\mathcal{G}^{(i)}$ is a \mathfrak{g} -submodule of \mathcal{L} isomorphic to $\mathfrak{g} \simeq \mathcal{G}$, $\mathcal{S}^{(j)}$ is a \mathfrak{g} -submodule isomorphic to \mathcal{S} and $\mathcal{V}^{(t)}$ is a \mathfrak{g} -submodule isomorphic to \mathcal{V} . Therefore as a vector space, we can identify \mathcal{L} with

$$(\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D}_0.$$

such that for each $\lambda \in \Lambda$, \mathcal{L}^λ is identified with

$$(\mathcal{G}^\lambda \dot{\otimes} \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot{\otimes} \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot{\otimes} \mathcal{C}) \oplus \mathcal{D}_0.$$

Now for $\lambda \in \Lambda$, $(\mathcal{L}^\lambda, \mathcal{L}^0)$ plays the same role as $(\mathcal{L}^n, \mathcal{L}^\ell)$ in §3.2.1 and so we are done using Step 3 together with Proposition 3.40. \square

Theorem 4.6. *Suppose that I is an infinite index set and I_0 is a subset of I of cardinality $\ell > 5$. Let R be an irreducible locally finite root system of type $X = D_I$ or $X = A_I$. Suppose that \mathcal{V} is a vector space with a basis $\{v_i \mid i \in I\}$ and take \mathcal{G} to be the finite dimensional split simple Lie algebra of type X as in Lemmas 2.5 or 2.10 respectively. Also define*

$$\mathfrak{I}_0 : \mathcal{V} \longrightarrow \mathcal{V} \quad v_i \mapsto \begin{cases} v_i & i \in I_0 \\ 0 & \text{otherwise.} \end{cases}$$

For $x, y \in \mathcal{G}$, define

$$x \circ y := xy + yx - \frac{2\text{tr}(xy)}{\ell + 1} \mathfrak{I}_0.$$

Suppose that $(\mathcal{A}, \text{id}_\mathcal{A}, \{0\}, \mathbf{0})$ is a coordinate quadrable of type X and \mathcal{K} is a subset of the full skew-dihedral homology group of \mathcal{A} satisfying the uniform property on \mathcal{A} . Set

$$\mathcal{L}(\mathcal{A}, \mathcal{K}) := (\mathcal{G} \otimes \mathcal{A}) \oplus \langle \mathcal{A}, \mathcal{A} \rangle,$$

in which $\langle \mathcal{A}, \mathcal{A} \rangle$ is the quotient space $\{\mathcal{A}, \mathcal{A}\}_\ell / \mathcal{K}$ (see Subsection 3.1) and for $a, a' \in \mathcal{A}$, take $\langle a, a' \rangle := \{a, a'\}_\ell + \mathcal{K}$, then $\mathcal{L}(\mathcal{A}, \mathcal{K})$ together with (4.7)

$$\begin{aligned} [x \otimes a, y \otimes a'] &= \begin{cases} [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + \text{tr}(xy)\langle a, a' \rangle & X = \dot{A}_I, \\ [x, y] \otimes aa' + \text{tr}(xy)\langle a, a' \rangle & X = D_I, \end{cases} \\ [\langle a_1, a_2 \rangle, x \otimes a] &= \begin{cases} -\frac{1}{2(\ell+1)}((x \circ \text{Id}_{\mathcal{V}_\ell}) \otimes [a, [a_1, a_2]]) \\ + [x, \text{Id}_{\mathcal{V}_\ell}] \otimes (a \circ [a_1, a_2]) + 2\text{tr}(\text{Id}_{\mathcal{V}_\ell} x)\langle a, [a_1, a_2] \rangle, & X = \dot{A}_I, \\ 0 & X = D_I, \end{cases} \\ [\langle a_1, a_2 \rangle, \langle a'_1, a'_2 \rangle] &= \begin{cases} \langle d_{a_1, a_2}^{\ell, \mathcal{A}}(a'_1), a'_2 \rangle + \langle a'_1, d_{a_1, a_2}^{\ell, \mathcal{A}}(a'_2) \rangle, & X = \dot{A}_I, \\ 0 & X = D_I, \end{cases} \end{aligned}$$

for $x, y \in \mathcal{G}$, $a, a', a_1, a_2, a'_1, a'_2 \in \mathcal{A}$, is a Lie algebra graded by R . Moreover any R -graded Lie algebra gives rise in this manner.

Theorem 4.8. Suppose that I is an infinite index set and I_0 is a subset of I of cardinality $\ell > 4$. Take \mathcal{G} to be either $\mathfrak{o}_B(I)$ or $\mathfrak{sp}(I)$. Suppose that \mathcal{V} is a vector space with a basis $\{v_0, v_i, v_{\bar{i}} \mid i \in I\}$ equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) as in (2.8) if $\mathcal{G} = \mathfrak{o}_B(I)$ and it is a vector space with a basis $\{v_i, v_{\bar{i}} \mid i \in I\}$ equipped with a nondegenerate skew-symmetric bilinear form (\cdot, \cdot) as in (2.13) if $\mathcal{G} := \mathfrak{sp}(I)$. Set

$$J := \begin{cases} I_0 \cup \bar{I}_0 \cup \{0\} & \text{if } \mathcal{G} = \mathfrak{o}_B(I) \\ I_0 \cup \bar{I}_0 & \text{if } \mathcal{G} = \mathfrak{sp}(I) \end{cases}$$

and define $\mathfrak{I}_0 : \mathcal{V} \longrightarrow \mathcal{V}$ to be the linear transformation defined by

$$v_i \mapsto \begin{cases} v_i & \text{if } i \in J \\ 0 & \text{if } i \in I \cup \bar{I} \setminus J. \end{cases}$$

Next set $\mathcal{S} := \mathcal{V}$ if $\mathcal{G} := \mathfrak{o}_B(I)$ and take \mathcal{S} to be as in (2.22) if $\mathcal{G} = \mathfrak{sp}(I)$. For $e, f \in \mathcal{G} \cup \mathcal{S}$, set

$$e \circ f := ef + fe - \frac{\text{tr}(ef)}{\ell} \mathfrak{I}_0.$$

Suppose that R is an irreducible locally finite root system of type $X = B_I$ or $X = C_I$ and $(\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadrable of type X . Take \mathcal{A} and \mathcal{B} to be the set of $*$ -fixed and $*$ -skew fixed points of \mathfrak{a} respectively. For a subset \mathcal{K} of the full skew-dihedral homology group of \mathfrak{a} satisfying the uniform property on \mathfrak{a} , set

$$\mathcal{L}(\mathfrak{a}, \mathcal{K}) := (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus \langle \mathfrak{a}, \mathfrak{a} \rangle,$$

in which $\langle \mathfrak{a}, \mathfrak{a} \rangle$ is the quotient space $\{\mathfrak{a}, \mathfrak{a}\}_\ell / \mathcal{K}$ (see Subsection 3.1), and for $\alpha, \alpha' \in \mathfrak{a}$, take $\langle \alpha, \alpha' \rangle := \{\alpha, \alpha'\} + \mathcal{K}$. Then $\mathcal{L}(\mathfrak{a}, \mathcal{K})$ together with

$$\begin{aligned}
 (4.9) \quad & [x \otimes a, y \otimes a'] = [x, y] \otimes aa' + \text{tr}(xy) \langle a, a' \rangle, \\
 & [x \otimes a, s \otimes b] = xs \otimes ab, \\
 & [s \otimes b, t \otimes b'] = D_{s,t} \otimes f(b, b') + (s, t) \langle b, b' \rangle \\
 & [\langle \alpha_1, \alpha_2 \rangle, x \otimes a] = x \otimes d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(a), \\
 & [\langle \alpha_1, \alpha_2 \rangle, s \otimes b] = s \otimes d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(b), \\
 & [\langle \alpha_1, \alpha_2 \rangle, \langle \alpha'_1, \alpha'_2 \rangle] = \langle d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(\alpha'_1), \alpha'_2 \rangle + \langle \alpha_1, d_{\alpha'_1, \alpha'_2}^{\ell, \mathfrak{a}}(\alpha_2) \rangle.
 \end{aligned}$$

(see Definition 2.6) for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $\alpha, \alpha', \alpha_1, \alpha_2, \alpha'_1, \alpha'_2 \in \mathfrak{a}$, if $\mathcal{G} = \mathfrak{o}_B(I)$ and

$$\begin{aligned}
 (4.10) \quad & [x \otimes a, y \otimes a'] = [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + \text{tr}(xy) \langle a, a' \rangle, \\
 & [x \otimes a, s \otimes b] = (x \circ s) \otimes \frac{1}{2}[a, b] + [x, s] \otimes \frac{1}{2}(a \circ b), \\
 & [s \otimes b, t \otimes b'] = [s, t] \otimes \frac{1}{2}b \circ b' + (s \circ t) \otimes \frac{1}{2}[b, b'] + \text{tr}(st) \langle b, b' \rangle, \\
 & [\langle \alpha, \alpha' \rangle, x \otimes a] = \frac{-1}{4\ell}((x \circ \mathfrak{I}_0) \otimes [a, \beta_{\alpha, \alpha'}^*] + [x, \mathfrak{I}_0] \otimes (a \circ \beta_{\alpha, \alpha'}^*)), \\
 & [\langle \alpha, \alpha' \rangle, s \otimes b] = \frac{-1}{4\ell}([s, \mathfrak{I}_0] \otimes (b \circ \beta_{\alpha_1, \alpha_2}^*) + (s \circ \mathfrak{I}_0) \otimes [b, \beta_{\alpha_1, \alpha_2}^*] + 2\text{tr}(s\mathfrak{I}_0) \langle b, \beta_{\alpha, \alpha'}^* \rangle), \\
 & [\langle \alpha_1, \alpha_2 \rangle, \langle \alpha'_1, \alpha'_2 \rangle] = \langle d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(\alpha'_1), \alpha'_2 \rangle + \langle \alpha'_1, d_{\alpha_1, \alpha_2}^{\ell, \mathfrak{a}}(\alpha'_2) \rangle.
 \end{aligned}$$

(see (3.7)) for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $\alpha, \alpha', \alpha_1, \alpha_2, \alpha'_1, \alpha'_2 \in \mathfrak{a}$, if $\mathcal{G} = \mathfrak{sp}(I)$ is a Lie algebra graded by R . Moreover any R -graded Lie algebra gives rise in this manner.

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School of Mathematics, Institute for Research in Fundamental Sciences
(IPM), P.O. Box: 19395-5746, Tehran, Iran.

Department of Mathematics, University of Isfahan, Isfahan, Iran, P.O.Box
81745-163.

ma.yousofzadeh@sci.ui.ac.ir